PERMUTATION GENERATION METHODS

Robert Sedgewick
Princeton University
Motivation

**PROBLEM** Generate all $N!$ permutations of $N$ elements

Q: Why?

- Basic research on a fundamental problem
- Compute exact answers for insights into combinatorial problems
- Structural basis for backtracking algorithms

Numerous published algorithms, dating back to 1650s

**CAVEATS**

- $N$ is between 10 and 20
- can be the basis for extremely dumb algorithms
- processing a perm often costs much more than generating it
N is between 10 and 20

<table>
<thead>
<tr>
<th>N</th>
<th>number of perms</th>
<th>million/sec</th>
<th>billion/sec</th>
<th>trillion/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3628800</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>39916800</td>
<td>seconds</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>479001600</td>
<td>minutes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>6227020800</td>
<td>hours</td>
<td>seconds</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>87178291200</td>
<td>day</td>
<td>minute</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>1307674368000</td>
<td>weeks</td>
<td>minutes</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>20922789888000</td>
<td>months</td>
<td>hours</td>
<td>seconds</td>
</tr>
<tr>
<td>17</td>
<td>355687428096000</td>
<td>years</td>
<td>days</td>
<td>minutes</td>
</tr>
<tr>
<td>18</td>
<td>6402373705728000</td>
<td>months</td>
<td>hours</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>121645100408832000</td>
<td>years</td>
<td>days</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>2432902008176640000</td>
<td>month</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

insignificant
impossible
Digression: analysis of graph algorithms

Typical graph-processing scenario:
- input graph as a sequence of edges (vertex pairs)
- build adjacency-lists representation
- run graph-processing algorithm

Q: Does the order of the edges in the input matter?
A: Of course!

Q: How?
A: It depends on the graph

Q: How?

There are $2^{V^2}$ graphs, so full employment for algorithm analysts
Digression (continued)

**Ex:** compute a spanning forest (DFS, stop when V vertices hit)

best case cost: V (right edge appears first on all lists)

Complete digraph on V vertices

\[ \text{worst case: } V^2 \]

\[ \text{average: } V \ln V \text{ (Kapidakis, 1990)} \]

Same graph with single outlier

\[ \text{worst case: } O(V^2) \]

\[ \text{average: } O(V^2) \]

**Can we estimate the average for a given graph?**

**Is there a simple way to reorder the edges to speed things up?**

**What impact does edge order have on other graph algorithms?**
Digression: analysis of graph algorithms

Insight needed, so generate perms to study graphs

No shortage of interesting graphs with fewer than 10 edges

Algorithm to compute average

- generate perms, run graph algorithm

Goal of analysis

- faster algorithm to compute average
Method 1: backtracking

Compute all perms of a global array by exchanging each element to the end, then recursively permuting the others

```c
exch (int i, int j)
{ int t = p[i]; p[i] = p[j]; p[j] = t; }
generate(int N)
{ int c;
  if (N == 1) doit();
  for (c = 1; c <= N; c++)
    { exch(c, N); generate(N-1); exch(c, N); } }
}
```

Invoke by calling
```
generate(N);
```

```
B C C D B D D C C A D A B D D A B A B C C A B A
C B D C D B C D A C A D D B A D A B D B A C A B
D D B B C C A A D D C C A A B B D D A A B B C C
A A A A A A B B B B B B C C C C C C D D D D D D
```

Problem: Too many (2N!) exchanges (!)
Method 2: “Plain changes”

Sweep first element back and forth to insert it into every position in each perm of the other elements

Generates all perms with N! exchanges of adjacent elements

Dates back to 1650s (bell ringing patterns in English churches)

Exercise: recursive implementation with constant time per exch
General single-exch recursive scheme

Eliminate first exch in backtracking

```c
exch (int i, int j)
    { int t = p[i]; p[i] = p[j]; p[j] = t; }
generate(int N)
    { int c;
        if (N == 1) doit();
        for (c = 1; c <= N; c++)
            { generate(N-1); exch(? , N); } }
```

**Detail(?):** Where is new item for \( p[N] \) each time?
Index table computation

Q: how do we find a new element for the end?
A: compute an index table from the (known) perm for N-1

so all perms of 3 takes into

so all perms of 4 takes into

and so forth

Exercise: Write a program to compute this table
Method 3: general recursive single-exch

Use precomputed index table

Generates perms with N! exchanges

Simple recursive algorithm

```c
int generate(int N)
{
    int c;
    if (N == 1) doit();
    for (c = 1; c <= N; c++)
    {
        generate(N-1);
        exch(B[N][c], N);
    }
}
```

No need to insist on particular sequence for last element

specifies \((N - 1)! (N - 2)! \ldots 3! 2!\) different algorithms

Table size is \(N(N-1)/2\) but \(N\) is less than 20

Do we need the table?
Method 4: Heap’s\* algorithm

Index table is not needed

Q: where can we find the next element to put at the end?
A: at \(1\) if \(N\) is odd; \(i\) if \(N\) is even

Exercise: Prove that it works!

*Note: no relationship between Heap and heap data structure
Simple recursive function

```c
int generate(int N)
{
    int c;
    if (N == 1) { doit(); return; }
    for (c = 1; c <= N; c++)
    {
        generate(N-1);
        exch(N % 2 ? 1 : c, N)
    }
}
```

N! exchanges
Starting point for code optimization techniques
Implementation of Heap's method (recursive)

Simple recursive function easily adapts to backtracking

```c
generate(int N)
{ int c;
  if (test(N)) return;
  for (c = 1; c <= N; c++)
  {
    generate(N-1);
    exch(N % 2 ? 1 : c, N)
  }
}
```

N! exchanges saved when test succeeds
Factorial counting

Count using a mixed-radix number system

```
for (n = 1; n <= N; n++)
    c[n] = 1;
for (n = 1; n <= N; )
    if (c[n] < n) { c[n]++; n = 1; }
    else c[n++] = 1;
```

Values of digit $i$ range from 1 to $i$

(Can derive code by systematic recursion removal)

1-1 correspondence with permutations

- commonly used to generate random perms

```
for (i = 1; i <=N i++) exch(i, random(i));
```

Use as control structure to generate perms
Implementation of Heap's method (nonrecursive)

```c
generate(int N) {
    int n, t, M;
    for (n = 1; n <= N; n++) {
        p[n] = n; c[n] = 1;
    }
    doit();
    for (n = 1; n <= N; ) {
        if (c[n] < n) {
            exch(N % 2 ? 1 : c, N)
            c[n]++; n = 1;
            doit();
        } else c[n++] = 1;
    }
}
```

"Plain changes" and most other algs also fit this schema
Analysis of Heap’s method

Most statements are executed $N!$ times (by design) except

- $B(N)$: the number of tests for $N$ equal to 1 (loop iterations)
- $A(N)$: the extra cost for $N$ odd

Recurrence for $B$

$$B(N) = NB(N-1) + 1 \quad \text{for } N > 1 \text{ with } B(1) = 1$$

Solve by dividing by $N!$ and telescoping

$$\frac{B(N)}{N!} = \frac{B(N-1)}{(N-1)!} + \frac{1}{N!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{N!}$$

Therefore $B(N) = \lfloor N! \cdot (e - 1) \rfloor$ and similarly $A(N) = \lceil N!/e \rceil$

Typical running time: $19N! + A(N) + 10B(N) \approx 36.55N!$

worthwhile to lower constant  

huge quantity
Improved version of Heap's method (recursive)

generate(int N)
{
    int c;
    if (N == 3)
    {
        doit();
        p1 = p[1]; p2 = p[2]; p3 = p[3];
    }
    for (c = 1; c <= N; c++)
    {
        generate(N-1);
        exch(N % 2 ? 1 : c, N)
    }
}
**Bottom line**

Quick empirical study on this machine (N = 12)

<table>
<thead>
<tr>
<th></th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heap (recursive)</td>
<td>415.2 secs</td>
</tr>
<tr>
<td>cc -O4</td>
<td>54.1 secs</td>
</tr>
<tr>
<td>Java</td>
<td>442.8 secs</td>
</tr>
<tr>
<td>Heap (nonrecursive)</td>
<td>84.0 secs</td>
</tr>
<tr>
<td>inline N = 2</td>
<td>92.4 secs</td>
</tr>
<tr>
<td>inline N = 3</td>
<td>51.7 secs</td>
</tr>
<tr>
<td>cc -O4</td>
<td>3.2 secs</td>
</tr>
</tbody>
</table>

about (1/6) billion perms/second

**Lower Bound:** about 2N! register transfers
References

Heap, “Permutations by interchanges,”
Computer Journal, 1963

Knuth, The Art of Computer Programming, vol. 4 sec. 7.2.1.1
//www-cs-faculty.stanford.edu/~knuth/taocp.html

Ord-Smith, “Generation of permutation sequences,”
Computer Journal, 1970-71

Sedgewick, Permutation Generation Methods,
Computing Surveys, 1977

Trotter, “Perm (Algorithm 115),”
CACM, 1962

Wells, Elements of combinatorial computing, 1961
[see surveys for many more]
Digression: analysis of graph algorithms

Initial results (Dagstuhl, 2002)