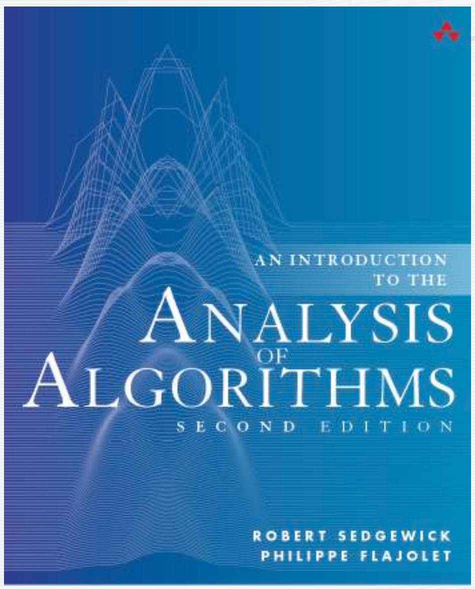

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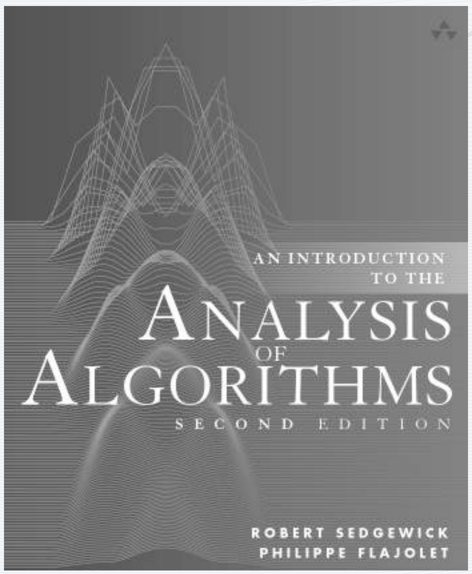
2. Recurrences



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2. Recurrences

- **Computing values**
- Telescoping
- Types of recurrences
- Mergesort
- Master Theorem

What is a recurrence?

Def. A *recurrence* is an equation that recursively defines a sequence.

Familiar example 1: *Fibonacci numbers*

recurrence

$$F_N = F_{N-1} + F_{N-2} \text{ for } N \geq 2 \text{ with } F_0 = 0 \text{ and } F_1 = 1$$

MUST specify for all N with initial conditions

sequence

`0, 1, 1, 2, 3, 5, 8, 13, 21, ...`

Q. Simple formula for sequence (function of N)?

What is a recurrence?

Recurrences directly model costs in programs.

Familiar example 2: *Quicksort* (see lecture 1)

recurrence

$$C_N = N + 1 + \sum_{0 \le k \le N-1} \frac{1}{N}(C_k + C_{N-k-1})$$

for $N \geq 1$ with $C_0 = 0$

sequence

0, 2, 5, 8 2/3, 12 5/6, 17 2/5,...

program

```
public class Quick
{
   private static int partition(Comparable[] a, int lo, int hi)
   {
      int i = lo, j = hi+1;
      while (true)
      {
         while (less(a[++i], a[lo])) if (i == hi) break;
         while (less(a[lo], a[--j])) if (j == lo) break;
         if (i >= j) break;
         exch(a, i, j);
      }
      exch(a, lo, j);
      return j;
   }

   private static void sort(Comparable[] a, int lo, int hi)
   {
      if (hi <= lo) return;
      int j = partition(a, lo, hi);
      sort(a, lo, j-1);
      sort(a, j+1, hi);
   }
}
```

Common-sense rule for solving any recurrence

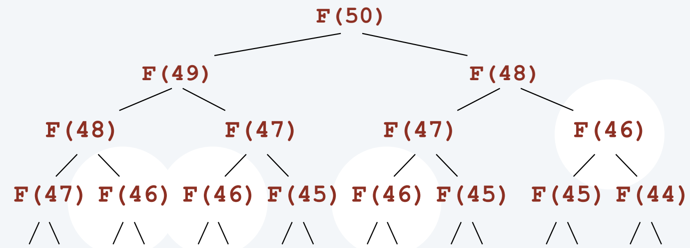
Use your computer to compute values. $F_N = F_{N-1} + F_{N-2}$ for $N \geq 2$ with $F_0 = 0$ and $F_1 = 1$

Use a recursive program?

```
public static void  F(int N)
{
    if (N == 0) return 0;
    if (N == 1) return 1;
    return F(N-1) + F(N-2);
}
```

✘

NO, NO, NO: Takes exponential time!



Top of recursion tree for naïve Fibonacci function

Instead, save all values in an array.

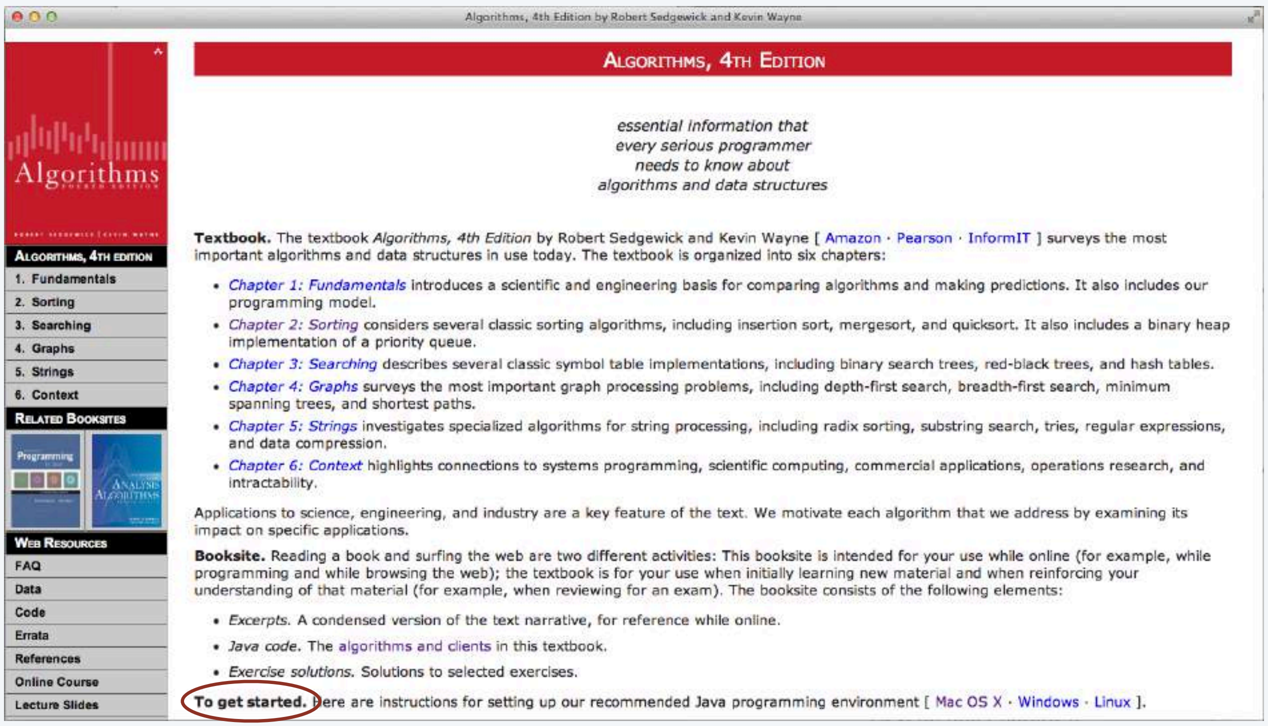
```
long[] F = new long[51];
F[0] = 0; F[1] = 1;
if (N == 1) return 1;
for (int i = 2; i <= 50; i++)
   F[i] = F[i-1] + F[i-2];
```

✓

Common-sense starting point for solving any recurrence

Use your computer to compute initial values.

First step: Download "standard model" from *Algorithms, 4th edition* booksite.



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`StdIn`	*Standard Input*
`StdOut`	*Standard Output*
`StdDraw`	*Standard Drawings*
`StdRandom`	*Random Numbers*
...	(Several other libraries)

Common-sense starting point for solving any recurrence

Use your computer to compute initial values (modern approach).

Ex. 1: *Fibonacci* $F_N = F_{N-1} + F_{N-2}$ with $F_0 = 0$ and $F_1 = 1$

Sequence.java

```
public interface Sequence
{
   public double eval(int N);
}
```

Fib.java

```
public class Fib implements Sequence
{
   private final double[] F;

   public Fib(int maxN)
   {
      F = new double[maxN+1];
      F[0] = 0; F[1]= 1;
      for (int N = 2; N <= maxN; N++)
         F[N] = F[N-1] + F[N-2];
   }

   public double eval(int N)
   {  return F[N];  }

   public static void main(String[] args)
   {
      int maxN = Integer.parseInt(args[0]);
      Fib F = new Fib(maxN);
      for (int i = 0; i < maxN; i++)
         StdOut.println(F.eval(i));
   }
}
```

Compute all values in the constructor

```
% java Fib 15
0.0
1.0
1.0
2.0
3.0
5.0
8.0
13.0
21.0
34.0
55.0
89.0
144.0
233.0
377.0
```

Common-sense starting point for solving any recurrence

Ex. 2: *Quicksort* $NC_N = (N+1)C_{N-1} + 2N$

QuickSeq.java

```
public class QuickSeq implements Sequence
{
   private final double[] c;

   public QuickSeq(int maxN)
   {
      c = new double[maxN+1];
      c[0] = 0;
      for (int N = 1; N <= maxN; N++)
          c[N] = (N+1)*c[N-1]/N + 2;
   }

   public double eval(int N)
   {  return c[N];  }

   public static void main(String[] args)
   {
      // Similar to Fib.java.
   }
}
```

```
% java QuickSeq 15
  0.000000
  2.000000
  5.000000
  8.666667
 12.833333
 17.400000
 22.300000
 27.485714
 32.921429
 38.579365
 44.437302
 50.477056
 56.683478
 63.043745
 69.546870
```

Common-sense starting point for solving any recurrence

Use your computer to plot initial values.

Values.java

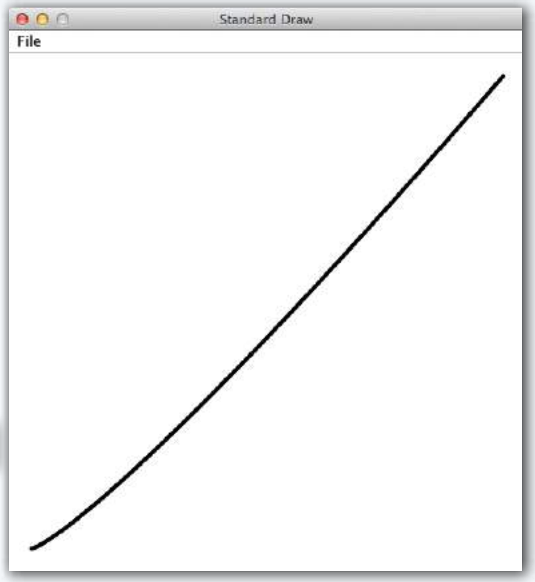
```
public class Values
{
   public static void show(Sequence f, int maxN)
   {
      double max = 0;
      for (int N = 0; N < maxN; N++)
          if (f.eval(N)>max) max = f.eval(N);
      for (int N = 0; N < maxN; N++)
      {
         double x = 1.0*N/maxN;
         double y = 1.0*f.eval(N)/max;
         StdDraw.filledCircle(x, y, .002);
      }
      StdDraw.show();
   }
}
```

QuickSeq.java

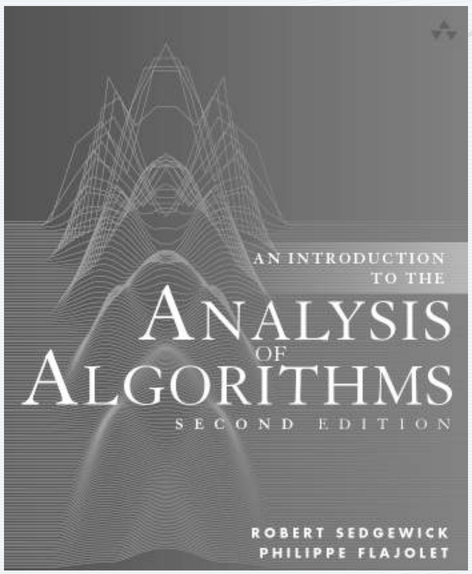
```
public class QuickSeq implements Sequence
{
   // Implementation as above.

   public static void main(String[] args)
   {
      int maxN = Integer.parseInt(args[0]);
      QuickSeq q = new QuickSeq(maxN);
      Values.show(q, maxN);
   }
}
```

```
% java QuickSeq 1000
```



ANALYTIC COMBINATORICS
PART ONE



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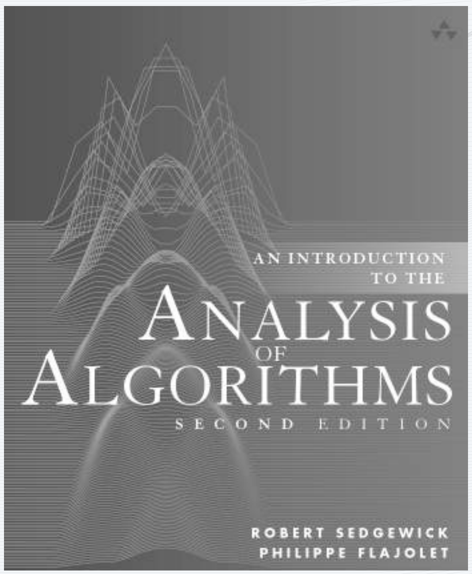
2. Recurrences

- **Computing values**
- Telescoping
- Types of recurrences
- Mergesort
- Master Theorem



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2. Recurrences

- Computing values
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- Types of recurrences
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- Master Theorem

Telescoping a (linear first-order) recurrence

Linear first-order recurrences *telescope* to a sum.

Example 1.

$$a_n = a_{n-1} + n \quad \textit{with} \;\; a_0 = 0$$

Apply equation for n–1

$$= a_{n-2} + (n-1) + n$$

Do it again

$$= a_{n-3} + (n-2) + (n-1) + n$$

Continue, leaving a sum

$$= a_0 + \sum_{1 \le k \le n} k$$

Evaluate sum

$$= \frac{(n+1)n}{2}$$

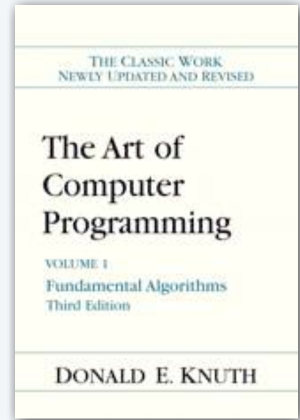
Check.

$$\frac{(n+1)n}{2} = \frac{n(n-1)}{2} + n$$

Challenge: Need to be able to evaluate the sum.

Elementary discrete sums

geometric series	$\displaystyle\sum_{0\le k<n} x^k = \frac{1-x^n}{1-x}$
arithmetic series	$\displaystyle\sum_{0\le k<n} k = \frac{n(n-1)}{2} = \binom{n}{2}$
binomial (upper)	$\displaystyle\sum_{0\le k\le n} \binom{k}{m} = \binom{n+1}{m+1}$
binomial theorem	$\displaystyle\sum_{0\le k\le n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$
Harmonic numbers	$\displaystyle\sum_{1\le k\le n} \frac{1}{k} = H_n$
Vandermonde convolution	$\displaystyle\sum_{0\le k\le n} \binom{n}{k}\binom{m}{t-k} = \binom{n+m}{t}$



see Knuth volume 1 for many more

Telescoping a (linear first-order) recurrence (continued)

When coefficients are not 1, multiply/divide by a *summation factor*.

Example 2.

$$a_n = 2a_{n-1} + 2^n \quad \textit{with} \;\; a_0 = 0$$

Divide by 2^n

$$\frac{a_n}{2^n} = \frac{a_{n-1}}{2^{n-1}} + 1$$

Telescope to a sum

$$\frac{a_n}{2^n} = \sum_{1 \le k \le n} 1 = n$$

$$a_n = n2^n$$

Check.

$$n2^n = 2(n-1)2^{n-1} + 2^n$$

Challenge: How do we find the summation factor?

Telescoping a (linear first-order) recurrence (continued)

Q. *What's the summation factor for* $a_n = x_n a_{n-1} + \ldots$?

A. Divide by $x_n x_{n-1} x_{n-2} \ldots x_1$

Example 3.

$$a_n = \left(1 + \frac{1}{n}\right) a_{n-1} + 2 \quad \textit{for } n > 0 \textit{ with } \; a_0 = 0$$

summation factor:

$$\frac{n+1}{n} \frac{n}{n-1} \frac{n-1}{n-2} \ldots \frac{3}{2} \frac{2}{1} = n+1$$

Divide by n+1

$$\frac{a_n}{n+1} = \frac{a_{n-1}}{n} + \frac{2}{n+1}$$

Telescope

$$= 2 \sum_{1 \le k \le n} \frac{1}{k+1} = 2H_{n+1} - 1$$

$$a_n = 2(n+1)(H_{n+1} - 1)$$

Challenge: Still need to be able to evaluate sums.

In-class exercise 1.

Verify the solution for *Example 3*.

Check initial values

$$a_n = \left(1 + \frac{1}{n}\right)a_{n-1} + 2 \quad \textit{for } n > 0 \textit{ with } \; a_0 = 0$$

$$a_1 = 2a_0 + 2 = 2$$

$$a_2 = \frac{3}{2}a_1 + 2 = 5$$

$$a_3 = \frac{4}{3}a_2 + 2 = 26/3$$

$$a_n = 2(n+1)(H_{n+1} - 1)$$

$$a_1 = 4(H_2 - 1) = 2$$

$$a_2 = 6(H_3 - 1) = 5$$

$$\begin{aligned} a_3 &= 8(H_4 - 1) \\ &= 8(1/2 + 1/3 + 1/4) \\ &= 26/3 \end{aligned}$$

Proof

$$\begin{aligned} (1 + \frac{1}{n})\overbrace{2n(H_n - 1)}^{a_{n-1}} + 2 \quad &= 2(n+1)(H_n - 1) + 2 \\ &= \underbrace{2(n+1)(H_{n+1} - 1)}_{a_n} \end{aligned}$$

In-class exercise 2.

Solve this recurrence:

$$na_n = (n-2)a_{n-1} + 2 \quad \textit{for } n > 1 \textit{ with } \; a_1 = 1$$

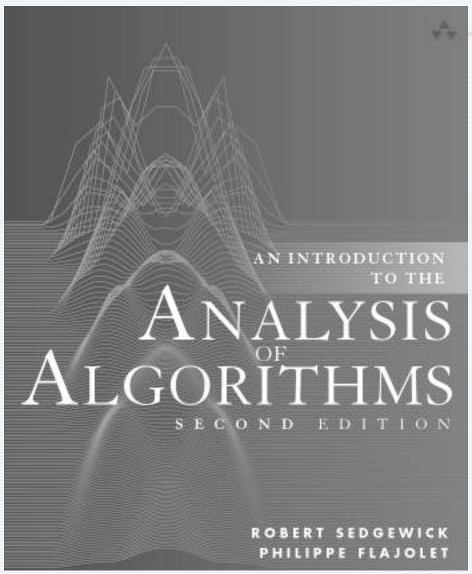
Hard way:

summation factor: $\dfrac{n-2}{n}\dfrac{n-3}{n-1}\dfrac{n-4}{n-2}\cdots = \dfrac{1}{n(n-1)}$

Easy way: $2a_2 = 2 \quad \text{so} \quad a_2 = 1$

$\text{therefore} \quad a_n = 1$

WHY?



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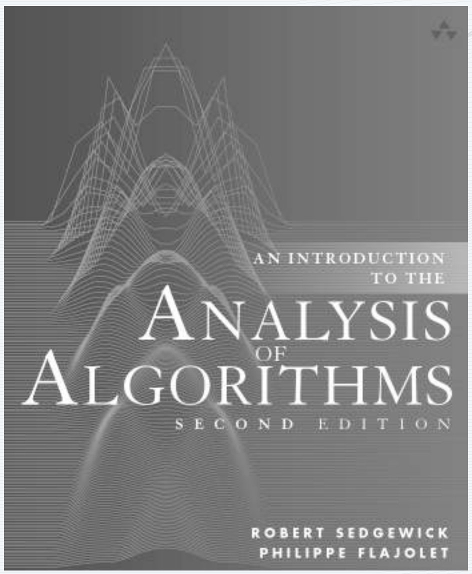
Recurrences

- Computing values
- **Telescoping**
- Types of recurrences
- Mergesort
- Master Theorem



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Recurrences

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Types of recurrences

first order	*linear*	$a_n = na_{n-1} - 1$
	nonlinear	$a_n = 1/(1 + a_{n-1})$
second order	*linear*	$a_n = a_{n-1} + 2a_{n-2}$
	nonlinear	$a_n = a_{n-1}a_{n-2} + \sqrt{a_{n-2}}$
	variable coefficients	$a_n = na_{n-1} + (n-1)a_{n-2} + 1$
higher order		$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-t})$
full history		$a_n = n + a_{n-1} + a_{n-2} \ldots + a_1$
divide-and-conquer		$a_n = a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} + n$

Nonlinear first-order recurrences

Example. (Newton's method)

$$c_N = \frac{1}{2}\left(c_{N-1} + \frac{2}{c_{N-1}}\right)$$

[Typical in scientific computing]

SqrtTwo.java

```
public class SqrtTwo implements Sequence
{
   private final double[] c;

   public SqrtTwo(int maxN)
   {
      c = new double[maxN+1];
      c[0] = 1;
      for (int N = 1; N <= maxN; N++)
            c[N] = (c[N-1] + 2/c[N-1])/2;
   }

   public double eval(int N)
   {  return c[N];  }

   public static void main(String[] args)
   {
      int maxN = Integer.parseInt(args[0]);
      SqrtTwo test = new SqrtTwo(maxN);
      for (int i = 0; i < maxN; i++)
         StdOut.println(test.eval(i));
   }
}
```

quadratic convergence: number of significant digits doubles for each iteration

```
% java SqrtTwo 10
1.0
1.5
1.4166666666666665
1.4142156862745097
1.4142135623746899
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095
1.414213562373095
```

Higher-order linear recurrences

[Stay tuned for systematic solution using generating functions (next lecture)]

Example 4.

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Postulate that $a_n = x^n$ $\qquad x^n = 5x^{n-1} - 6x^{n-2}$

Divide by x^{n-2} $\qquad x^2 - 5x + 6 = 0$

Factor $\qquad (x-2)(x-3) = 0$

Form of solution must be $\qquad a_n = c_0 3^n + c_1 2^n$

Use initial conditions to solve for coefficients

$$a_0 = 0 = c_0 + c_1$$

$$a_1 = 1 = 3c_0 + 2c_1$$

Note dependence on initial conditions

Solution is $c_0 = 1$ and $c_1 = -1$ $\qquad a_n = 3^n - 2^n$

Higher-order linear recurrences

[Stay tuned for systematic solution using generating functions (next lecture)]

Example 5. Fibonacci numbers

$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Postulate that $a_n = x^n$ $\qquad x^n = x^{n-1} + x^{n-2}$

Divide by x^{n-2} $\qquad x^2 - x - 1 = 0$

Factor $\qquad (x - \phi)(x - \hat{\phi}) = 0$

$$\phi = \frac{1 + \sqrt{5}}{2} \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

Form of solution must be $\qquad a_n = c_0\phi^n + c_1\hat{\phi}^n$

Use initial conditions to solve for coefficients

$$a_0 = 0 = c_0 + c_1$$

$$a_1 = 1 = \phi c_0 + \hat{\phi} c_1$$

Note dependence on initial conditions

Solution $\qquad a_n = \dfrac{\phi^n}{\sqrt{5}} - \dfrac{\hat{\phi}^n}{\sqrt{5}}$

Higher-order linear recurrences (continued)

Procedure amounts to an *algorithm*.

Example 5. Fibonacci numbers

$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0 \text{ and } a_1 = 1$$

Postulate that $a_n = x^n$	$x^n = x^{n-1} + x^{n-2}$
Divide by x^{n-2}	$x^2 - x - 1 = 0$
Factor	$(x - \phi)(x - \hat{\phi}) = 0$
Form of solution must be	$a_n = c_0\phi^n + c_1\hat{\phi}^n$
Use initial conditions to solve for coefficients	$a_0 = 0 = c_0 + c_1$ $a_1 = 1 = \phi c_0 + \hat{\phi} c_1$
Solution	$a_n = \dfrac{\phi^n}{\sqrt{5}} - \dfrac{\hat{\phi}^n}{\sqrt{5}}$

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

Note dependence on initial conditions

Multiple roots? Add $n\alpha^n$ terms (see text)

Need to compute roots? Use symbolic math package.

```
sage: realpoly.<z> = PolynomialRing(CC)
sage: factor(z^2-z-1)
(z - 1.61803398874989) * (z + 0.618033988749895)
```

Complex roots? Stay tuned for systematic solution using GFs (next lecture)

Divide-and-conquer recurrences

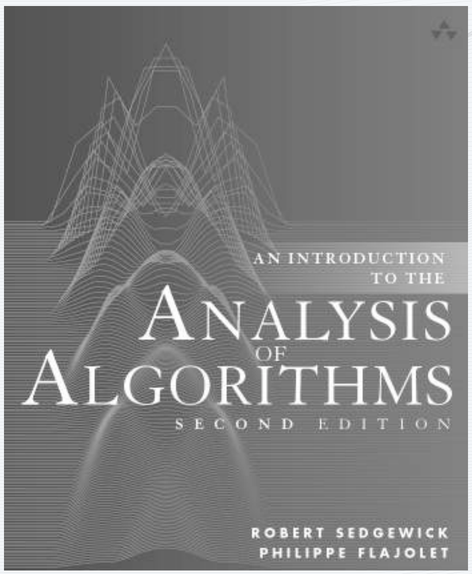
Divide and conquer is an effective technique in algorithm design.

Recursive programs map directly to recurrences.

Classic examples:

- Binary search
- Mergesort
- Batcher network
- Karatsuba multiplication
- Strassen matrix multiplication

ANALYTIC COMBINATORICS
PART ONE



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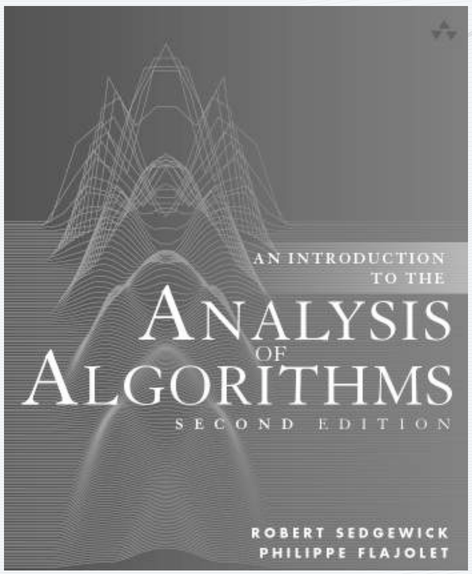

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Recurrences

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Warmup: binary search

Everyone's first divide-and-conquer algorithm

```
// Precondition: array a[] is sorted.
public static int rank(int key, int[] a)
{
   int lo = 0;
   int hi = a.length - 1;
   while (lo <= hi)
   {
      // Key is in a[lo..hi] or not present.
      int mid = lo + (hi - lo) / 2;
      if      (key < a[mid]) hi = mid - 1;
      else if (key > a[mid]) lo = mid + 1;
      else return mid;
   }
   return -1;
}
```

Number of compares in the worst case

$$B_N = B_{\lfloor N/2 \rfloor} + 1 \quad \textit{for } N > 1 \textit{ with } \; B_1 = 1$$

Analysis of binary search (easy case)

$$B_N = B_{\lfloor N/2 \rfloor} + 1 \quad \textit{for } N > 1 \textit{ with } \ B_1 = 1$$

Exact solution for $N = 2^n$.

$$a_n \equiv B_{2^n}$$

$$a_n = a_{n-1} + 1 \quad \textit{for } n > 0 \textit{ with } \ a_0 = 1$$

Telescope to a sum

a0 +

+ 1

$$a_n = \sum_{1 \le k \le n} 1 = n$$

+ 1

$$B_N = \lg N \quad \textit{when N is a power of 2}$$

Check. $\lg N = \lg(N/2) + 1$

Analysis of binary search (general case)

Easy by correspondence with binary numbers

Define B_N to be the number of bits in the binary representation of N.

- $B_1 = 1$.
- Removing the rightmost bit of N gives $\lfloor N/2 \rfloor$.

Therefore $B_N = B_{\lfloor N/2 \rfloor} + 1 \quad \textit{for } N > 1 \textit{ with } B_1 = 1$

same recurrence as for binary search

Example.

1101011	110101	1
107	53	
N	$\lfloor N/2 \rfloor$	

Theorem. $B_N = \lfloor \lg N \rfloor + 1$

Proof. Immediate by definition of $\lfloor\ \rfloor$.

$$B_N = n + 1 \quad \text{for } 2^n \le N < 2^{n+1}$$

$$\text{or } \; n \le \lg N < n + 1 \implies n = \lfloor \lg N \rfloor$$

N	1	2	3	4	5	6	7	8	9
binary	1	10	11	100	101	110	111	1000	1001
$\lg N$	0	1.0	1.58...	2.0	2.32...	2.58...	2.80...	3	3.16...
$\lfloor \lg N \rfloor$	0	1	1	2	2	2	2	3	3
$\lfloor \lg N \rfloor + 1$	1	2	2	3	3	3	3	4	4

Mergesort

Everyone's *second* divide-and-conquer algorithm

```
public class Merge
{
   ...
   private static void
   sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
       if (hi <= lo) return;
       int mid = lo + (hi - lo) / 2;
       sort(a, aux, lo, mid);
       sort(a, aux, mid + 1, hi);
       merge(a, aux, lo, mid, hi);
   }
   ...
}
```

For simplicity, assume merge implementation uses N compares

Number of compares for sort: $C_N = C_{\lfloor N/2 \rfloor} + C_{\lceil N/2 \rceil} + N \quad \textit{for } N > 1 \textit{ with } \; C_1 = 1$

Analysis of mergesort (easy case)

Number of compares for sort: $C_N = C_{\lfloor N/2 \rfloor} + C_{\lceil N/2 \rceil} + N \quad \textit{for } N > 1 \textit{ with } \; C_1 = 1$

Already solved for $N = 2^n$

Example 2.

$$a_n = 2a_{n-1} + 2^n \quad \textit{with} \;\; a_0 = 0$$

Divide by 2^n

$$\frac{a_n}{2^n} = \frac{a_{n-1}}{2^{n-1}} + 1$$

Telescope to a sum

$$\frac{a_n}{2^n} = \sum_{1 \le k \le n} 1 = n$$

$$a_n = n2^n$$

Solution: $C_N = N\lg N \quad \textit{when } N \textit{ is a power of 2}$

Analysis of mergesort (general case)

Number of compares for sort: $C_N = C_{\lfloor N/2 \rfloor} + C_{\lceil N/2 \rceil} + N \quad \textit{for } N > 1 \textit{ with } \ C_1 = 1$ 0

Solution: $C_N = N \lg N \quad \textit{when } N \textit{ is a power of } 2$

Q. For quicksort, the number of compares is $\sim 2N \ln N - 2(1-\gamma)N$

Is the number of compares for mergesort $\sim N \lg N + \alpha N$ for some constant α?

A. NO !

Coefficient of the linear term for mergesort

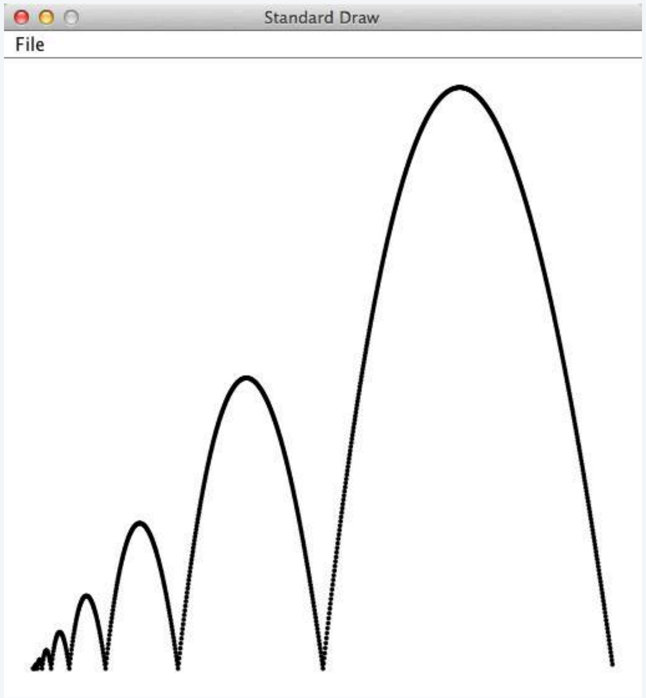
```
public class MergeLinearTerm implements Sequence
{
   private final double[] c;

   public MergeLinear(int maxN)
   {
      c = new double[maxN+1];
      c[0] = 0;
      for (int N = 1; N <= maxN; N++)
         c[N] = N + c[N/2] + c[N-(N/2)];
      for (int N = 1; N <= maxN; N++)
          c[N] -= N*Math.log(N)/Math.log(2)) + N;
   }

   public double eval(int N)
   {  return c[N];  }

   public static void main(String[] args)
   {
      int maxN = Integer.parseInt(args[0]);
      MergeLinearTerm M = new MergeLinearTerm(maxN);
      Values.show(M, maxN);
   }
}
```

```
% java MergeLinearTerm 512
```



Analysis of mergesort (general case)

Number of compares for sort: $C_N = C_{\lfloor N/2 \rfloor} + C_{\lceil N/2 \rceil} + N \quad \textit{for } N > 1 \textit{ with } \; C_1 = 1$ 0

Same formula for N+1.

$$C_{N+1} = C_{\lfloor (N+1)/2 \rfloor} + C_{\lceil (N+1)/2 \rceil} + N + 1$$
$$= C_{\lceil N/2 \rceil} + C_{\lfloor N/2 \rfloor + 1} + N + 1$$

Subtract.

$$C_{N+1} - C_N = C_{\lfloor N/2 \rfloor + 1} - C_{\lfloor N/2 \rfloor} + 1$$

Define $D_N = C_{N+1} - C_N$.

$$D_N = D_{\lfloor N/2 \rfloor} + 1$$

Solve as for binary search.

$$D_N = \lfloor \lg N \rfloor + 2$$

different initial value

Telescope.

$$C_N = N - 1 + \sum_{1 \le k < N} (\lfloor \lg k \rfloor + 1)$$

$\lceil N/2 \rceil = \lfloor (N+1)/2 \rfloor$

1	0	1	1	1
2	1	1	1	2
3	1	2	2	2
4	2	2	2	3
5	2	3	3	3
6	3	3	3	4
7	3	4	4	4
8	4	4	4	5
9	4	5	5	5

$\lfloor N/2 \rfloor + 1 = \lceil (N+1)/2 \rceil$

Theorem. $C_N = N - 1$ + *number of bits in binary representation of numbers* $< N$

Combinatorial correspondence

S_N = number of bits in the binary rep. of all numbers < N

	=	$S_{\lfloor N/2 \rfloor}$	+	$S_{\lceil N/2 \rceil}$	+	$N-1$
1		1		1		1
10		10		10		10
11		11		11		11
100		100		100		100
101		101		101		101
110		110		110		110
111		111		111		111
1000		1000		1000		1000
1001		1001		1001		1001
1010		1010		1010		1010
1011		1011		1011		1011
1100		1100		1100		1100
1101		1101		1101		1101
1110		1110		1110		1110

$$S_N = S_{\lfloor N/2 \rfloor} + S_{\lceil N/2 \rceil} + N - 1$$

Same recurrence as mergesort (except for –1): $C_N = S_N + N - 1$

Number of bits in all numbers < N (alternate view)

8 4 2 1

0000
0001
0010
0011
0100
0101
0110
0111
1000
1001
1010
1011
1100
1101
1110

N

$\lfloor \lg N \rfloor + 1$

bits are in an N by ⌊lgN+1⌋ box

subtract off red 0s column by column

$$S_N = N(\lfloor \lg N \rfloor + 1) - \sum_{0 \le k \le \lfloor \lg N \rfloor} 2^k$$
$$= N\lfloor \lg N \rfloor - 2^{\lfloor \lg N \rfloor + 1} + N + 1$$

$$C_N = S_N + N - 1$$
$$= N\lfloor \lg N \rfloor - 2^{\lfloor \lg N \rfloor + 1} + 2N$$

Theorem. Number of compares for mergesort is $N\lfloor \lg N \rfloor - 2^{\lfloor \lg N \rfloor + 1} + 2N$

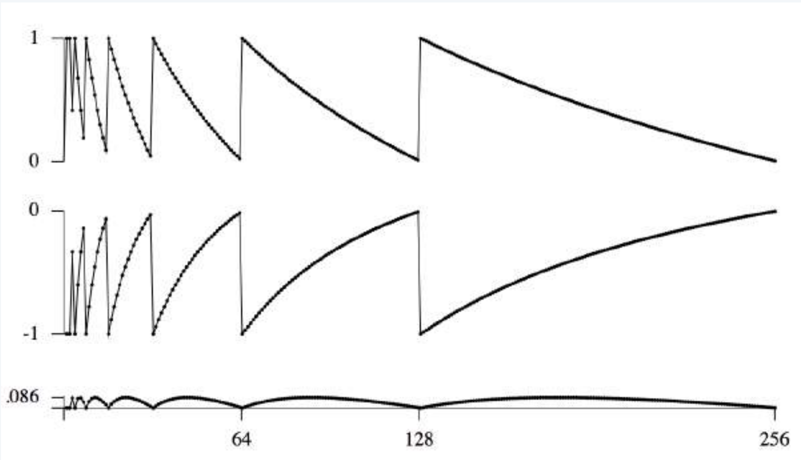
Analysis of mergesort (summary)

Number of compares for sort: $C_N = C_{\lfloor N/2 \rfloor} + C_{\lceil N/2 \rceil} + N$ *for* $N > 1$ *with* $C_1 = 1$ 0

Solution: $C_N = N \lg N$ *when N is a power of 2*

Theorem. Number of compares for mergesort is $N\lfloor \lg N \rfloor - 2^{\lfloor \lg N \rfloor + 1} + 2N$

Alternate formulation (Knuth). $C_N = N \lg N + N\alpha(N)$



$\alpha(N)$

Notation: $\lfloor \lg N \rfloor = \lg N - \{\lg N\}$

$1 - \{\lg N\}$

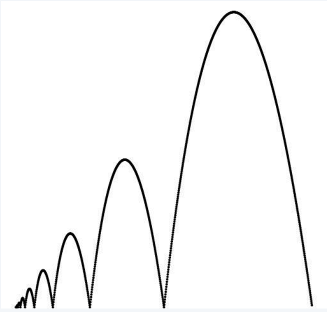
$+$

$1 - 2^{1 - \{\lg N\}}$

$=$

$2 - \{\lg N\} - 2^{1 - \{\lg N\}}$

$N\alpha(N)$



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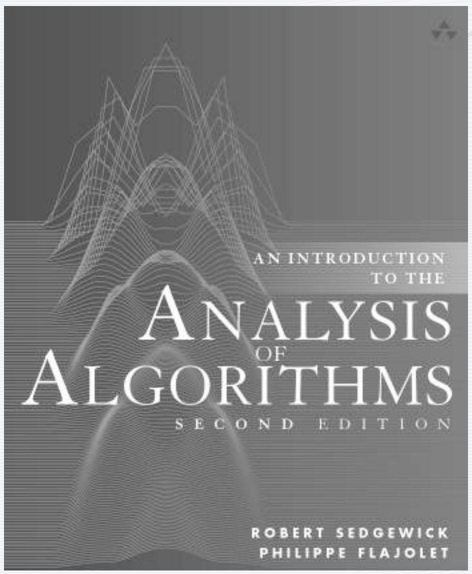

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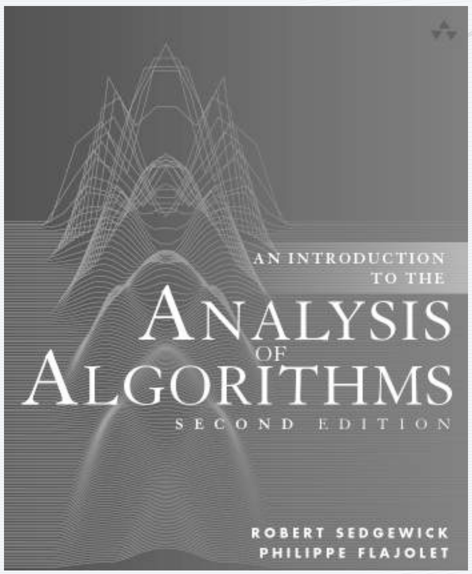
Recurrences

- Computing values
- Telescoping
- Types of recurrences
- **Mergesort**
- Master Theorem



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Recurrences

- Computing values
- Telescoping
- Types of recurrences
- Mergesort
- **Master Theorem**

Divide-and-conquer algorithms

Suppose that an algorithm attacks a problem of size N by

- Dividing into α parts of size about N/β.
- Solving recursively.
- Combining solutions with extra cost $\Theta(N^{\gamma}(\log N)^{\delta})$

only valid when
N is a power of 2

Example 1 (mergesort): $\alpha = 2$, $\beta = 2$, $\gamma = 1$, $\delta = 0$

$$C_N = 2C_{N/2} + N$$

Example 2 (Batcher network): $\alpha = 2$, $\beta = 2$, $\gamma = 1$, $\delta = 1$

$$C_N = 2C_{N/2} + N \lg N$$

Example 3 (Karatsuba multiplication): $\alpha = 3$, $\beta = 2$, $\gamma = 1$, $\delta = 0$

$$C_N = 3C_{N/2} + N$$

Example 4 (Strassen matrix multiply): $\alpha = 7$, $\beta = 2$, $\gamma = 1$, $\delta = 0$

$$C_N = 7C_{N/2} + N$$

“Master Theorem” for divide-and-conquer algorithms

Suppose that an algorithm attacks a problem of size n by
dividing into α parts of size about n/β with extra cost $\Theta(n^\gamma(\log n)^\delta)$

Theorem. The solution to the recurrence

$$a_n = \underbrace{a_{n/\beta+O(1)} + a_{n/\beta+O(1)} + \ldots +}_{\alpha \text{ terms}} a_{n/\beta+O(1)} + \Theta(n^\gamma(\log n)^\delta)$$

is given by

$$a_n = \Theta(n^\gamma(\log n)^\delta) \qquad \text{when } \gamma < \log_\beta \alpha$$

$$a_n = \Theta(n^\gamma(\log n)^{\delta+1}) \qquad \text{when } \gamma = \log_\beta \alpha$$

$$a_n = \Theta(n^{\log_\beta \alpha}) \qquad \text{when } \gamma > \log_\beta \alpha$$

>

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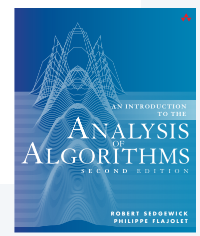
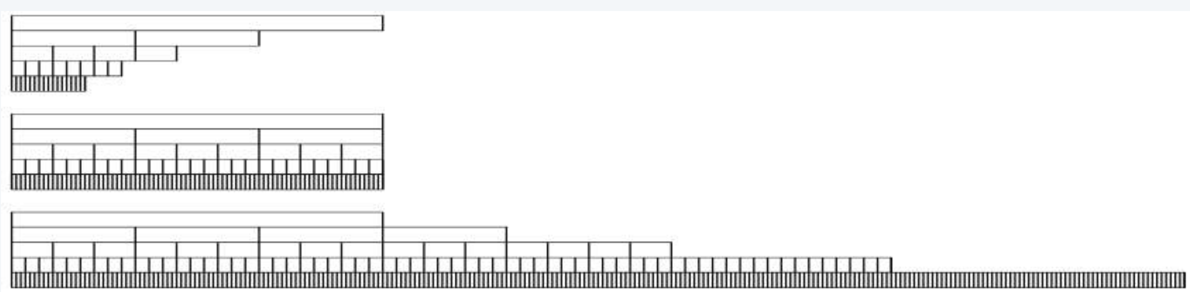


Example: α = 3

β = 2

β = 3

β = 4



Typical "Master Theorem" applications

Suppose that an algorithm attacks a problem of size N by

- Dividing into α parts of size about N/β.
- Solving recursively.
- Combining solutions with extra cost $\Theta(N^{\gamma}(\log N)^{\delta})$

Master Theorem

$a_n = \Theta(n^{\gamma}(\log n)^{\delta})$	when $\gamma < \log_{\beta} \alpha$
$a_n = \Theta(n^{\gamma}(\log n)^{\delta+1})$	when $\gamma = \log_{\beta} \alpha$
$a_n = \Theta(n^{\log_{\beta} \alpha})$	when $\gamma > \log_{\beta} \alpha$

Asymptotic growth rate

Example 1 (mergesort): $\alpha = 2$, $\beta = 2$, $\gamma = 1$, $\delta = 0$ — $\Theta(N \log N)$

Example 2 (Batcher network): $\alpha = 2$, $\beta = 2$, $\gamma = 1$, $\delta = 1$ — $\Theta(N(\log N)^2)$

Example 3 (Karatsuba multiplication): $\alpha = 3$, $\beta = 2$, $\gamma = 1$, $\delta = 0$ — $\Theta(N^{\log_2 3}) = \Theta(N^{1.585\ldots})$

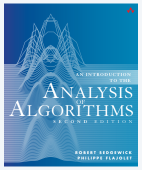
Example 4 (Strassen matrix multiply): $\alpha = 7$, $\beta = 2$, $\gamma = 1$, $\delta = 0$ — $\Theta(N^{\log_2 7}) = \Theta(N^{2.807\ldots})$

Versions of the "Master Theorem"

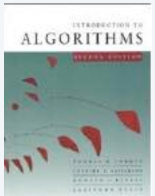
Suppose that an algorithm attacks a problem of size N by

- Dividing into α parts of size about N/β.
- Solving recursively.
- Combining solutions with extra cost $\Theta(N^{\gamma}(\log N)^{\delta})$

1. Precise results are available for certain applications in the analysis of algorithms.

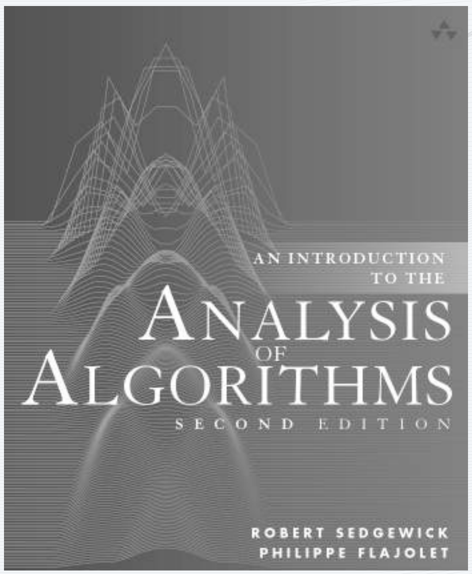


2. General results are available for proofs in the theory of algorithms.



3. A full solution using analytic combinatorics was derived in 2011 by Szpankowski and Drmota.

see "A Master Theorem for Divide-and-Conquer Recurrences" by Szpankowski and Drmota (SODA 2011).



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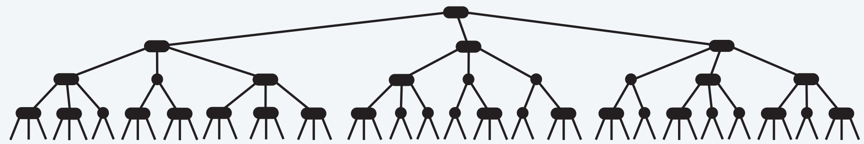
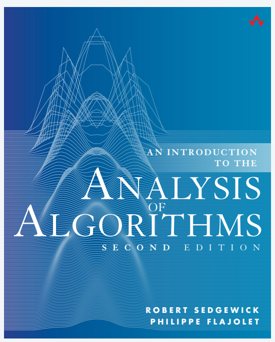
Recurrences

- Computing values
- Telescoping
- Types of recurrences
- Mergesort
- **Master Theorem**

Exercise 2.17

Percentage of three nodes at the bottom level of a 2-3 tree?





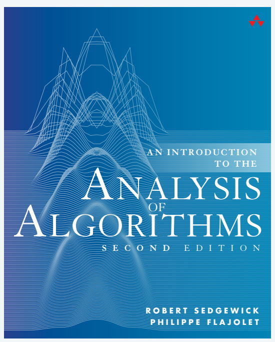
Exercise 2.17 [Yao] ("Fringe analysis of 2–3 trees") Solve the recurrence

$$A_N = A_{N-1} - \frac{2A_{N-1}}{N} + 2\Big(1 - \frac{2A_{N-1}}{N}\Big) \qquad \text{for } N > 0 \text{ with } A_0 = 0.$$

This recurrence describes the following random process: A set of N elements collect into "2-nodes" and "3-nodes." At each step each 2-node is likely to turn into a 3-node with probability $2/N$ and each 3-node is likely to turn into two 2-nodes with probability $3/N$. What is the average number of 2-nodes after N steps?

Exercise 2.69

Details of divide-by-three and conquer?



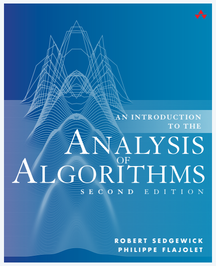
Exercise 2.69 Plot the periodic part of the solution to the recurrence

$$a_N = 3a_{\lfloor N/3 \rfloor} + N \qquad \text{for } N > 3 \text{ with } a_1 = a_2 = a_3 = 1$$

for $1 \leq N \leq 972$.

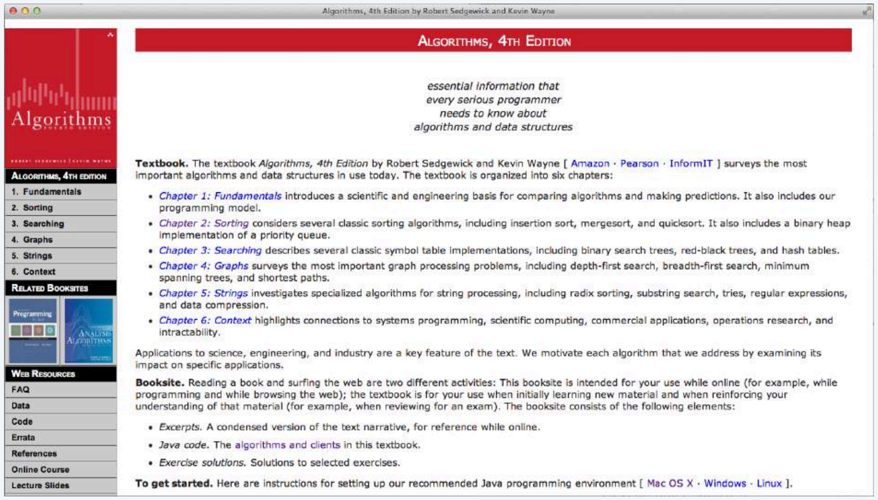
Assignments for next lecture

1. Read pages 41-86 in text.



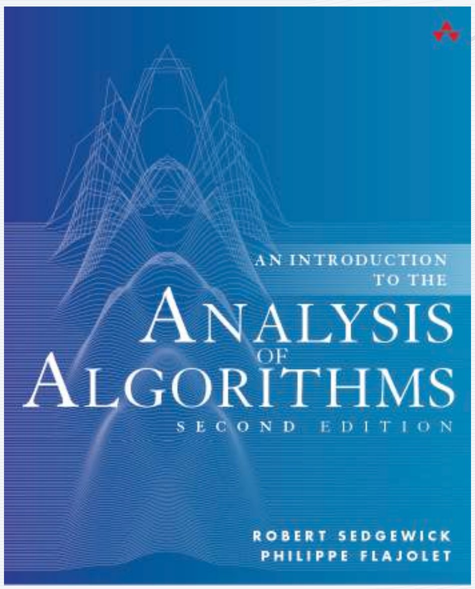
2. Write up solution to Ex. 2.17.

3. Set up `StdDraw` from *Algs* booksite



4. Do Exercise 2.69.

ANALYTIC COMBINATORICS
PART ONE



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2. Recurrences