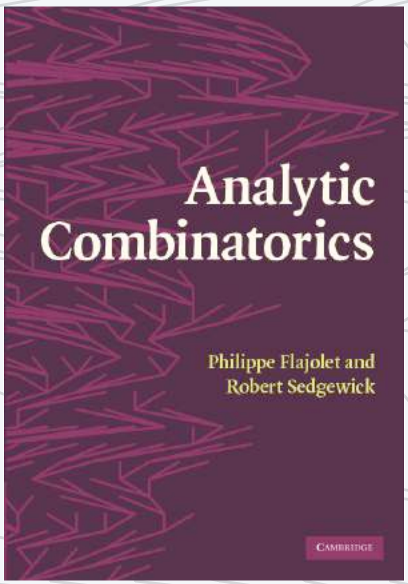
ANALYTIC COMBINATORICS

PART TWO



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1. Combinatorial structures and OGFs

Attention: Much of this lecture is a *quick review* of material in *Analytic Combinatorics, Part I*

One consequence: it is a bit longer than usual

To: Students who took *Analytic Combinatorics, Part I*

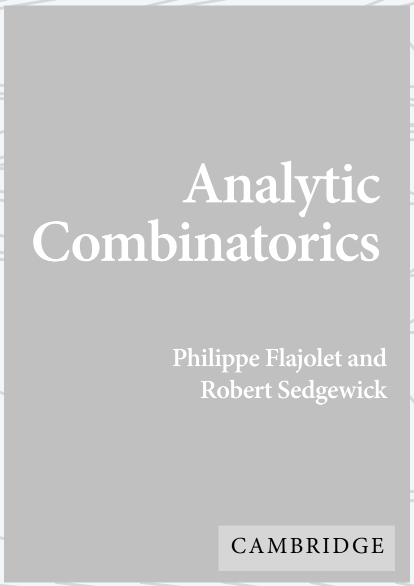
Bored because you understand it all?

GREAT! Skip to the section on labelled trees and do the exercises.

To: Students starting with *Analytic Combinatorics, Part II*

Moving too fast? Want to see details and motivating applications?

No problem, watch Lectures 5, 6, and 8 in Part I.



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1. Combinatorial structures and OGFs

- **Symbolic method**
- Trees and strings
- Powersets and multisets
- Compositions and partitions
- Substitution

Analytic combinatorics overview

To analyze properties of a large combinatorial structure:

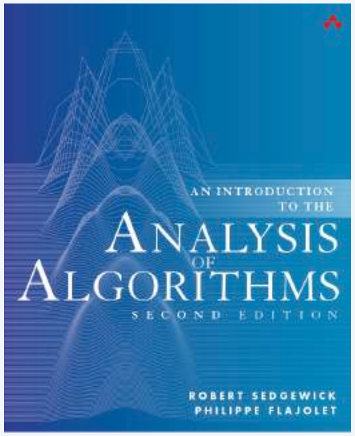
1. Use the symbolic method
   - Define a *class* of combinatorial objects
   - Define a notion of *size* (and associated generating function)
   - Use standard operations to develop a *specification* of the structure

Result: A direct derivation of a GF equation (implicit or explicit)

Classic next steps:

- Extract coefficients
- Use classic asymptotics to estimate coefficients

Result: Asymptotic estimates that quantify the desired properties



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See *An Introduction to the Analysis of Algorithms* for a gentle introduction

Analytic combinatorics overview

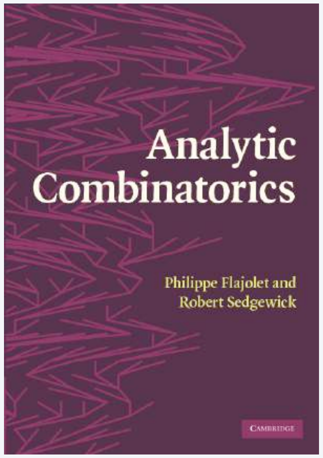
To analyze properties of a large combinatorial structure:

1. Use the symbolic method
   - Define a *class* of combinatorial objects.
   - Define a notion of *size* (and associated generating function)
   - Use standard operations to develop a *specification* of the structure.

Result: A direct derivation of a GF equation (implicit or explicit).

2. Use complex asymptotics to estimate growth of coefficients.
   - [no need for explicit solution]
   - [stay tuned for details]

Result: Asymptotic estimates that quantify the desired properties



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See *Analytic Combinatorics* for a rigorous treatment

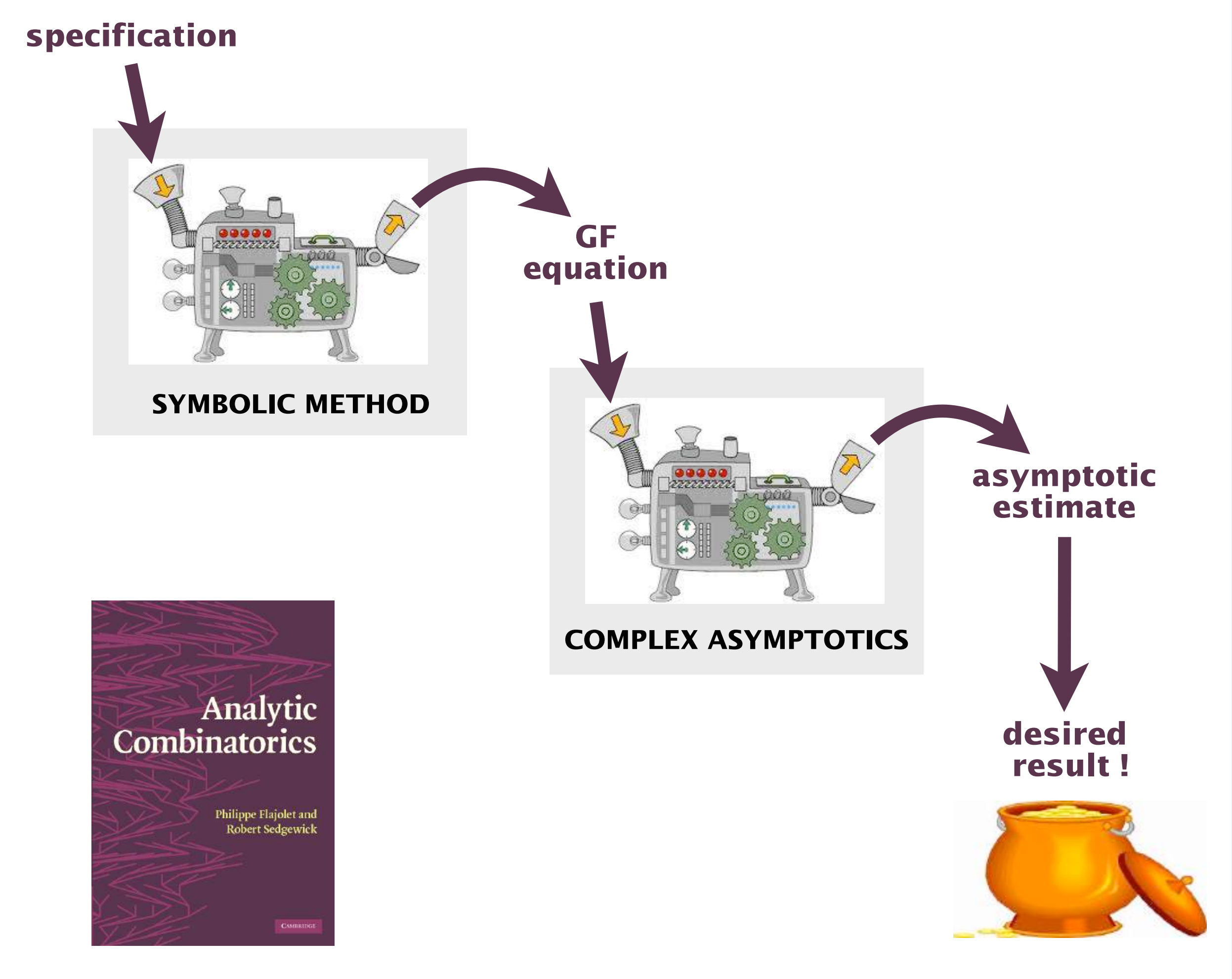
Analytic combinatorics overview

A. SYMBOLIC METHOD

1. OGFs
2. EGFs
3. MGFs

B. COMPLEX ASYMPTOTICS

4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point



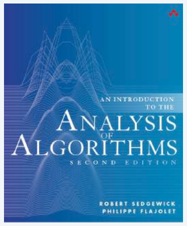
The symbolic method

An approach for *directly* deriving GF equations.

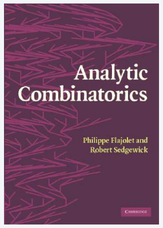
- Define a *class* of combinatorial objects.
- Define a notion of *size* (and associated generating function)
- Define *operations* suitable for constructive definitions of objects.
- Prove *correspondences* between operations and GFs.

Result: A GF equation (implicit or explicit).

See *An Introduction to the Analysis of Algorithms* for a gentle introduction



See *Analytic Combinatorics* for a rigorous treatment



This lecture: An overview that assumes *some* familiarity. ← Ex: Part I of this course

Basic definitions

Def. A *combinatorial class* is a set of combinatorial objects and an associated *size* function.

Def. The *ordinary generating function* (OGF) associated with a class is the formal power series $A(z) = \sum_{a \in A} z^{|a|}$

← size function (|a|); object name (a); class name (A)

Fundamental (elementary) identity

$$A(z) \equiv \sum_{a \in A} z^{|a|} = \sum_{N \geq 0} A_N z^N$$

Q. How many objects of size N ?

A. $A_N = [z^N]A(z)$

Fantasy: Different letter for each class

Reality: Only 26 letters!

Usual conventions

class name	roman	A
OGF name	roman with arg	$A(z)$
object variable	lowercase	a
coefficient	subscripted	A_N
size	N or n	

With the symbolic method, we specify the class *and at the same time* characterize the OGF

Unlabeled classes: cast of characters

TREES
Recursive structures
T_N = [*Catalan #s*]

STRINGS
Sequences of characters
$S_N = N^M$

INTEGERS
N objects
$I_N = 1$

COMPOSITIONS
Positive integers sum to N
$C_N = 2^{N-1}$

LANGUAGES
Sets of strings
[REs and CFGs]

PARTITIONS
Unordered compositions
[enumeration not elementary]

The symbolic method (basic constructs)

Suppose that A and B are classes of unlabeled objects with enumerating OGFs $A(z)$ and $B(z)$.

operation	notation	semantics	OGF
disjoint union	$A + B$	disjoint copies of objects from A and B	$A(z) + B(z)$
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from A and one from B	$A(z)B(z)$
sequence	$SEQ(A)$	sequences of objects from A	$\frac{1}{1 - A(z)}$

Stay tuned for other constructs

Proofs of correspondences

A + *B*

$$\sum_{c\in A+B} z^{|c|} = \sum_{a\in A} z^{|a|} + \sum_{b\in B} z^{|b|} = A(z) + B(z)$$

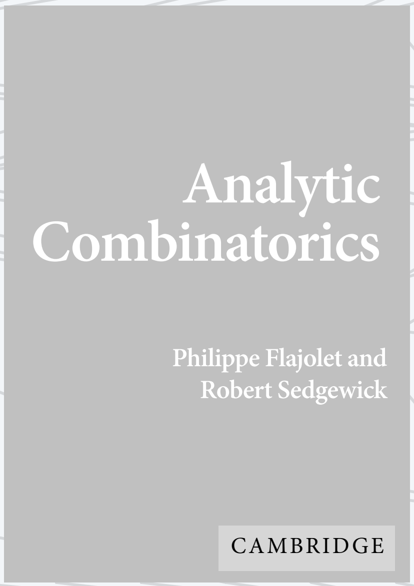
A × *B*

$$\sum_{c\in a\times b} z^{|c|} = \sum_{a\in A}\sum_{b\in B} z^{|a|+|b|} = \Big(\sum_{a\in A} z^{|a|}\Big)\Big(\sum_{b\in B} z^{|b|}\Big) = A(z)B(z)$$

Text

SEQ(*A*)

construction	OGF
$SEQ_k(A) \equiv A^k$	$A(z)^k$
$SEQ_T(A) \equiv A^{t_1} + A^{t_2} + A^{t_3} + \ldots$ where $T \equiv t_1, t_2, t_3, \ldots$ is a subset of the integers	$A(z)^{t_1} + A(z)^{t_2} + A(z)^{t_3} + \ldots$
$SEQ(A) \equiv \epsilon + A + A^2 + A^3 + \ldots$	$1 + A(z) + A(z)^2 + A(z)^3 + \ldots = \dfrac{1}{1 - A(z)}$



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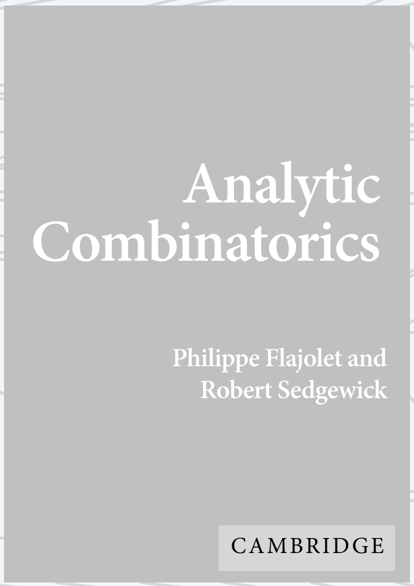
1. Combinatorial structures and OGFs

- **Symbolic method**
- Trees and strings
- Powersets and multisets
- Compositions and partitions
- Substitution



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1. Combinatorial structures and OGFs

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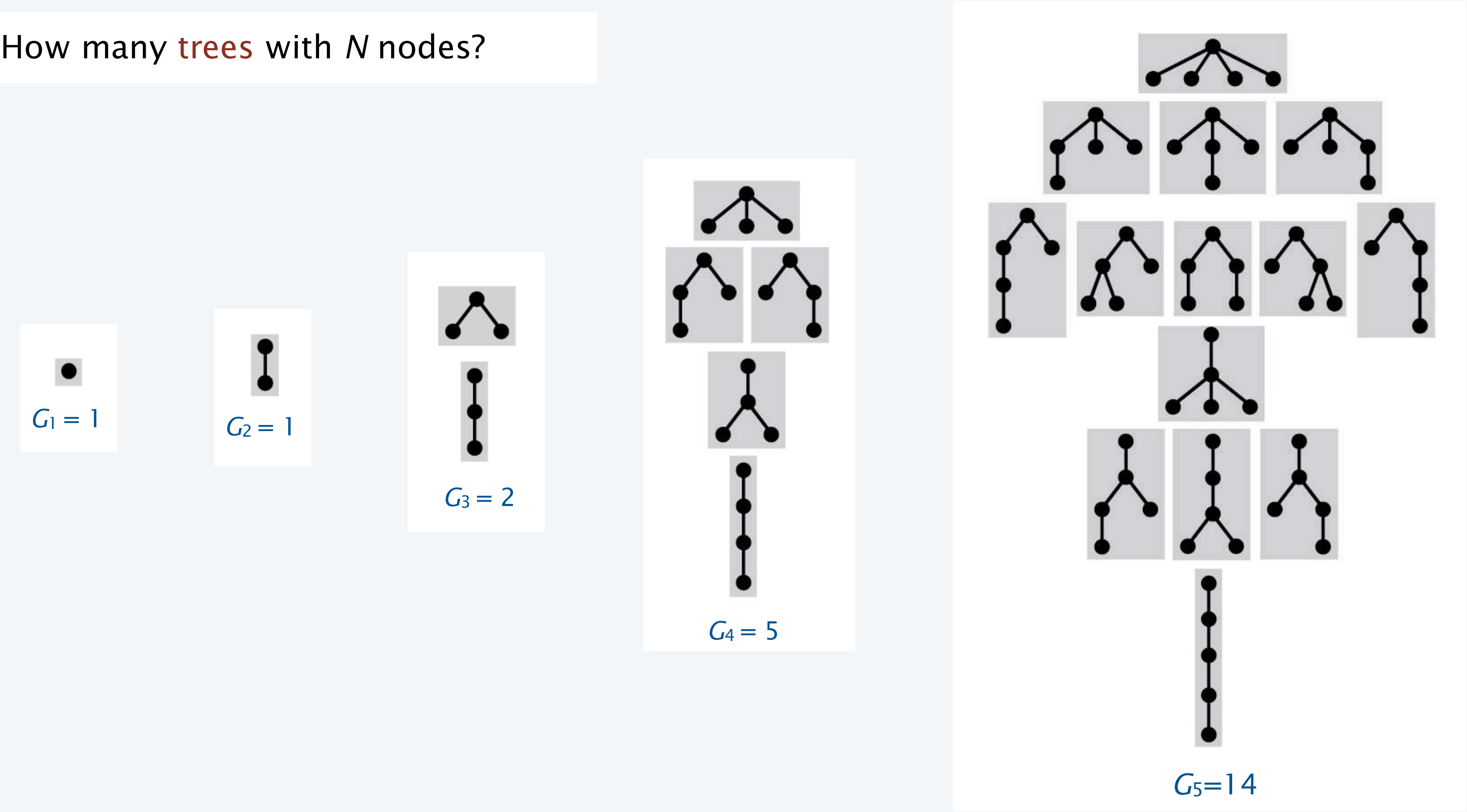
Classic example of the symbolic method

Q. How many trees with N nodes?



Analytic combinatorics: How many trees with N nodes?

Symbolic method

Combinatorial class — G, the class of all trees

Construction — $G = \bullet \times SEQ(G)$ ← "a tree is a node and a sequence of trees"

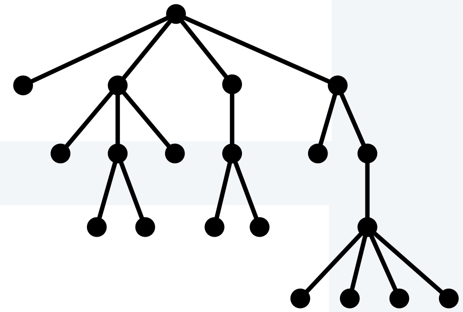
OGF equation

$$G(z) = z(1 + G(z) + G(z)^2 + G(z)^3 + \ldots) = \frac{z}{1 - G(z)}$$

$$G(z) - G(z)^2 = z$$

Quadratic equation

$$G(z) = \frac{1 + \sqrt{1 - 4z}}{2}$$



Classic next steps

Binomial theorem

$$G(z) = -\frac{1}{2}\sum_{N\geq 1}\binom{\frac{1}{2}}{N}(-4z)^N$$

Extract coefficients

$$G_N = -\frac{1}{2}\binom{\frac{1}{2}}{N}(-4)^N = \frac{1}{N}\binom{2N-2}{N-1} \quad = \frac{1}{4N-2}\binom{2N}{N}$$

Stirling's approximation

$$\sim \frac{1}{4N}\exp\big(2N\ln(2N) - 2N + \ln\sqrt{4\pi N} - 2(N\ln(N) - N + \ln\sqrt{2\pi N})\big)$$

Simplify

$$G_N \sim \frac{4^{N-1}}{\sqrt{\pi N^3}}$$

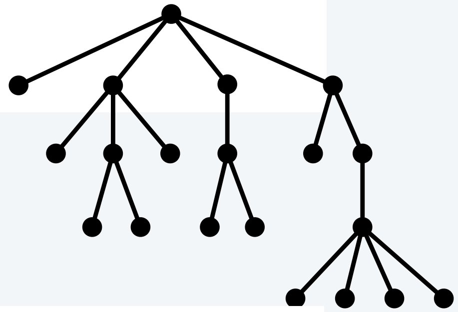
detailed calculations omitted

Analytic combinatorics: How many trees with N nodes?

Symbolic method

Combinatorial class	G, the class of all trees
Construction	$G = \bullet \times SEQ(G)$
OGF equation	$G(z) = z(1 + G(z) + G(z)^2 + G(z)^3 + \ldots) = \dfrac{z}{1 - G(z)}$
	$G(z) - G(z)^2 = z$

"a tree is a node and a sequence of trees"



Complex asymptotics

Singularity analysis	$G_N = [z^N]G(z) \sim \dfrac{4^N}{\Gamma(1/2)\sqrt{N}} = \dfrac{4^N}{\sqrt{\pi N}}$

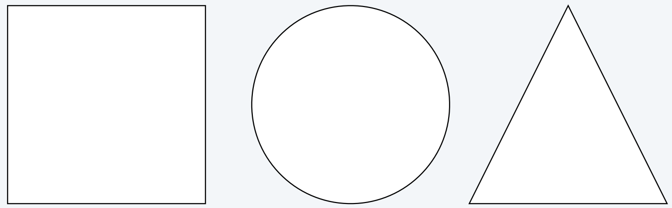
GF equation *directly* implies asymptotics

This lecture: Focus on symbolic method for deriving OGF equations (stay tuned for asymptotics).

A standard paradigm for the symbolic method

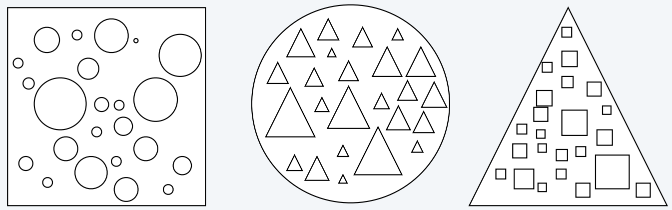
Fundamental constructs

- elementary or trivial
- confirm intuition



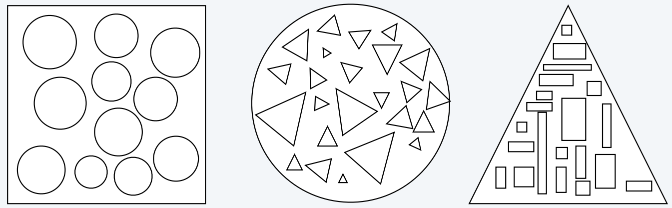
Compound constructs

- many possibilities
- classical combinatorial objects
- expose underlying structure
- one of many paths to known results



Variations

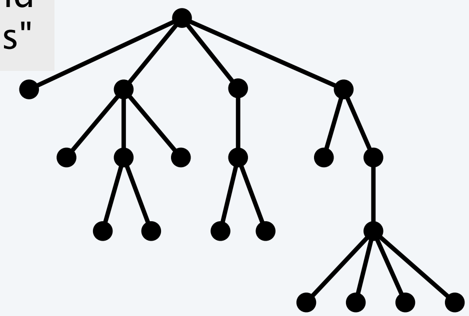
- unlimited possibilities
- *not* easily analyzed otherwise



Variations on a theme 1: Trees

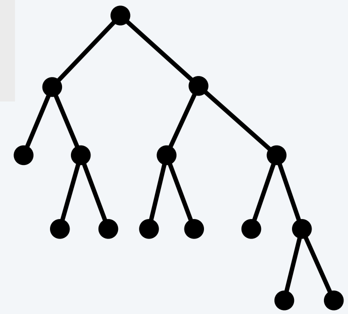
Fundamental construct

Combinatorial class	G, the class of all trees	
Construction	$G = \bullet \times SEQ(G)$	"a tree is a node and a sequence of trees"
OGF equation	$G(z) = z(1 + G(z) + G(z)^2 + G(z)^3 + \ldots) = \dfrac{z}{1 - G(z)}$	
	$G(z) - G(z)^2 = z$	



Variation on the theme: *restrict each node to* 0 *or* 2 *children*

Combinatorial class	T, the class of binary trees	
Construction	$T = \bullet \times SEQ_{0,2}(T)$	"a binary tree is a node and a sequence of 0 or 2 binary trees"
OGF equation	$T(z) = z(1 + T(z)^2)$	

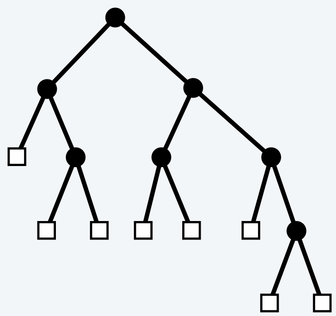


Variations on a theme 1: Trees (continued)

Variation on the theme: multiple node types

Combinatorial class	$T^\bullet$, binary trees, *enumerated by internal nodes*
Atoms	(see table below)
Construction	$T = \square + T \times \bullet \times T$
OGF equation	$T^\bullet(z) = 1 + zT^\bullet(z)^2$

type	*class*	*size*	*GF*
external node	$\square$	0	1
internal node	$\bullet$	1	z



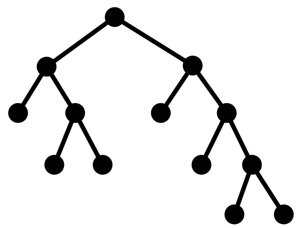
Combinatorial class	$T^\bullet$, binary trees, *enumerated by external nodes*
OGF equation	$T^\square(z) = z + T^\square(z)^2$

More variations: unary-binary trees, ternary trees, ...

Still more variations: gambler's ruin sequences, context-free languages, triangulations, ...

Some variations on ordered (rooted plane) trees

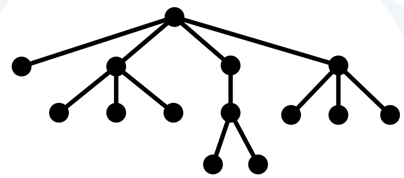
Binary



$T = \bullet \times SEQ_{0,2}(T)$

$T(z) = z(1 + T(z)^2)$

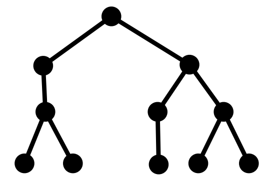
Ordered



$G = \bullet \times SEQ(G)$

$$G(z) = \frac{z}{1 - G(z)}$$

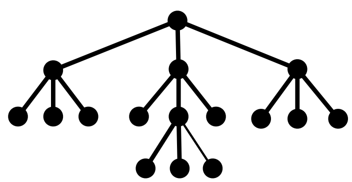
Unary-binary



$M = \bullet \times SEQ_{\leq 2}(M)$

$M(z) = z(1 + M(z) + M(z)^2)$

Ternary



$T = \bullet \times SEQ_{0,3}(T)$

$T(z) = z(1 + T(z)^3)$

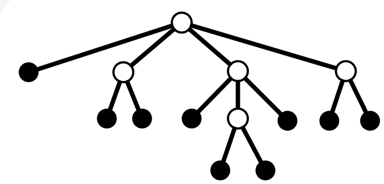
Arbitrary restrictions

$T = \bullet \times SEQ_{\Omega}(T)$

$T^{\Omega}(z) = z\phi(T^{\Omega}(z))$

$$\phi(u) \equiv \sum_{\omega \in \Omega} u^{\omega}$$

Bracketings



$S = \bullet + SEQ_{\geq 2}(S)$

$$S(z) = z + \frac{S(z)^2}{1 - S(z)}$$

Variation on a theme 2: Strings

Fundamental construct

Combinatorial class	B, the class of all binary strings	
Construction	$B = E + (Z_0 + Z_1) \times B$	"a binary string is empty or a bit followed by a binary string"
OGF equation	$B(z) = 1 + 2zB(z)$	

Variation on the theme: *disallow sequences of P or more 0s*

Combinatorial class	B_P, the class of all binary strings with no 0^P	
Construction	$B_P = Z_{<P}(E + Z_1 B_P)$	"a string with no 0^P is a string of 0s of length <P followed by an empty string or a 1 followed by a string with no 0^P"
OGF equation	$B_P(z) = (1 + z + \ldots + z^P)(1 + zB_P(z))$	

More variations: disallow any pattern (autocorrelation), REs, CFGs ...

Some variations on strings

M-ary

$$B = SEQ(Z_0 + \ldots + Z_{M-1})$$

$$B(z) = \frac{1}{1 - Mz}$$

Binary

$$B = E + (Z_0 + Z_1) \times B$$
$$B = SEQ(Z_0 + Z_1)$$

$$B(z) = \frac{1}{1 - 2z}$$

Exclude 0^P

$$B_P = Z_{<P}(E + Z_1 \times B_P)$$

$$B_P(z) = \frac{1 - z^P}{1 - 2z + z^{P+1}}$$

Regular languages

[Rational OGFs]

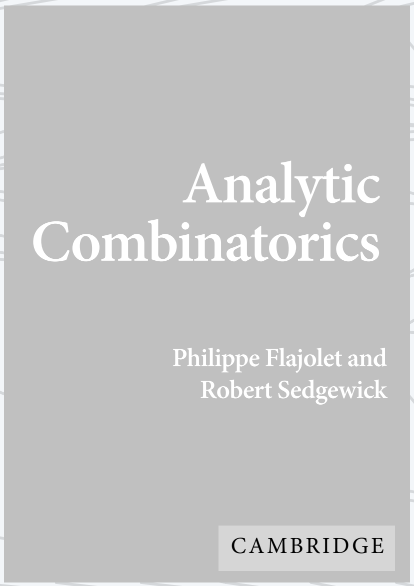
Context-free languages

[Algebraic OGFs]

Exclude pattern p

$$S_p(z) = \frac{c_p(z)}{z^P + (1 - 2z)c_p(z)}$$

[See Part I, Lecture 8]



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1. Combinatorial structures and OGFs

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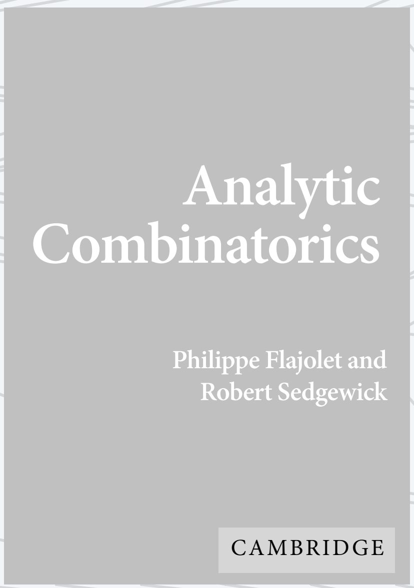
- **Trees and strings**
- Powersets and multisets
- Compositions and partitions
- Substitution



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1. Combinatorial structures and OGFs

- Symbolic method
- Trees and strings
- **Powersets and multisets**
- Compositions and partitions
- Substitution

The symbolic method (two additional constructs)

Suppose that A is a class of unlabeled objects with enumerating OGF $A(z)$.

operation	notation	semantics	OGF
powerset	$PSET(A)$	finite sets of objects from A (no repetitions)	[stay tuned]
multiset	$MSET(A)$	finite sets of objects from A (with repetitions)	[stay tuned]

Powersets

Def. The *powerset* of a class A is the class consisting of all subsets of A.

PSET { a }

{ }
{ a }

$P_1 = 2$

PSET { a, b }

{ }
{ a }
{ b }
{ a, b }

$P_2 = 4$

PSET { a, b, c }

{ }
{ a }
{ b }
{ a, b }
{ c }
{ a, c }
{ b, c }
{ a, b, c }

$P_3 = 8$

PSET { a, b, c, d }

subsets without d	same subsets with d
{ }	{ d }
{ a }	{ a, d }
{ b }	{ b, d }
{ a, b }	{ a, b, d }
{ c }	{ c, d }
{ a, c }	{ a, c, d }
{ b, c }	{ b, c, d }
{ a, b, c }	{ a, b, c, d }

$P_4 = 16$

Lemma: PSET $\{a_1, a_2, \ldots a_M\}$ = PSET $\{a_1, a_2, \ldots a_{M-1}\} \times (\{\} + \{a_M\})$

Powersets

Atoms

notation	size	GF
a_k	1	z

Combinatorial class: P_M, the powerset class for M atoms

Example: {a, c, f, g, h}

OGF: $$P_M(z) = \sum_{p \in P_M} z^{|p|} = \sum_{N \geq 0} P_{MN} z^N$$ ← P_{MN} is the # of subsets of size N (no repetitions)

Construction: $P_M = (\{\} + \{a_1\}) \times (\{\} + \{a_2\}) \times \ldots \times (\{\} + \{a_M\})$

OGF equation: $$P_M(z) = (1+z)^M$$

Expansion: $$P_{MN} = \binom{M}{N}$$ ✓

$$P_M(1) = 2^M$$ ✓

↑ total # subsets of M atoms

Multisets

Def. The *multiset* of a class A is the class consisting of all subsets of A *with repetitions allowed*.

MSET {*a*}

```
{}
{a}
{a, a}
{a, a, a}
...
```

MSET {*a*, *b* }

```
{}
{a}
{a, a}
{a, a, a}

{b}
{a, b}
{a, a, b}
{a, a, a, b}

{b, b}
{a, b, b}
{a, a, b, b}
{a, a, a, b, b}
```

MSET {*a*, *b*, *c* }

```
{}                {c}                   {c, c}
{a}               {a, c}                {a, c, c}
{a, a}            {a, a, c}             {a, a, c, c}
{a, a, a}         {a, a, a, c}          {a, a, a, c, c}

{b}               {b, c}                {b, c, c}
{a, b}            {a, b, c}             {a, b, c, c}
{a, a, b}         {a, a, b, c}          {a, a, b, c, c}
{a, a, a, b}      {a, a, a, b, c}       {a, a, a, b, c, c}

{b, b}            {b, b, b, c}          {b, b, c, c}
{a, b, b}         {a, b, b, b, c}       {a, b, b, c, c}
{a, a, b, b}      {a, a, b, b, b, c}    {a, a, b, b, c, c}
{a, a, a, b, b}   {a, a, a, b, b, b, c} {a, a, a, b, b, c, c}
```

Lemma: MSET $\{a_1, a_2, \ldots a_M\}$ = MSET $\{a_1, a_2, \ldots a_{M-1}\} \times$ SEQ $\{a_M\}$

Multisets

Atoms

notation	*size*	*GF*
a_k	1	z

Combinatorial class — S_M, the multiset class for M atoms

Example — `{a, a, a, b, b, b, c}`

OGF — $$S_M(z) = \sum_{s \in S_M} z^{|s|} = \sum_{N \geq 0} S_{MN} z^N$$ ← S_{MN} is the # of subsets of size N (with repetitions)

Construction — $$S_M = SEQ(a_1) \times SEQ(a_2) \times \ldots \times SEQ(a_M)$$

OGF equation — $$S_M(z) = \frac{1}{(1-z)^M}$$

Expansion — $$S_{MN} = \binom{N+M-1}{M-1}$$ ✓

The symbolic method (two additional constructs)

Suppose that A is a class of unlabeled objects with enumerating OGF $A(z)$.

operation	notation	semantics	OGF
powerset	$PSET(A)$	finite sets of objects from A (no repetitions)	$\prod_{n\geq 1}(1+z^n)^{A_n} = \exp\Big(-\sum_{k\geq 1}\frac{(-1)^k A(z^k)}{k}\Big)$
multiset	$MSET(A)$	finite sets of objects from A (with repetitions)	$\prod_{n\geq 1}\frac{1}{(1-z^n)^{A_n}} = \exp\Big(\sum_{k\geq 1}\frac{A(z^k)}{k}\Big)$

Proof of correspondences for powersets

PSET(*A*)

construction	OGF
$PSET(\{a,b\}) = \big(\{\} + \{a\}\big) \times \big(\{\} + \{b\}\big)$	$(1+z^{\lvert a \rvert})(1+z^{\lvert b \rvert})$
$PSET(A) \equiv \prod_{a \in A} \big(\{\} + \{a\}\big)$	$\prod_{a \in \mathcal{A}} (1+z^{\lvert a \rvert}) = \prod_{N \geq 0} (1+z^N)^{A_N}$

exp-log version

$$\prod_{N\geq 0}(1+z^N)^{A_N} = \exp\Big(\sum_{N\geq 0} A_N \ln(1+z^N)\Big)$$

$$= \exp\Big(-\sum_{N\geq 0} A_N \sum_{k\geq 1}(-1)^k \frac{z^{Nk}}{k}\Big)$$

$$= \exp\Big(-\sum_{k\geq 1}(-1)^k \frac{A(z^k)}{k}\Big)$$

$$= \exp\Big(A(z) - \frac{A(z^2)}{2} + \frac{A(z^3)}{3} - \ldots\Big)$$

Proof of correspondences for multisets

$MSET(A)$

construction	OGF
$MSET(\{a,b\}) = SEQ(\{a\}) \times SEQ(\{b\})$	$\frac{1}{(1-z^{\lvert a \rvert})(1-z^{\lvert b \rvert})}$
$MSET(A) \equiv \prod_{a \in A} SEQ(\{a\})$	$\prod_{a \in A} \frac{1}{(1-z^{\lvert a \rvert})} = \prod_{N \geq 0} \frac{1}{(1-z^N)^{A_N}}$

exp-log version

$$\prod_{N \geq 0} \frac{1}{(1-z^N)^{A_N}} = \exp\Big(\sum_{N \geq 0} A_N \ln \frac{1}{1-z^N}\Big)$$

$$= \exp\Big(\sum_{N \geq 0} A_N \sum_{k \geq 1} \frac{z^{Nk}}{k}\Big)$$

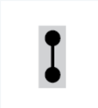
$$= \exp\Big(\sum_{k \geq 1} \frac{A(z^k)}{k}\Big)$$

$$= \exp\Big(A(z) + \frac{A(z^2)}{2} + \frac{A(z^3)}{3} + \ldots\Big)$$

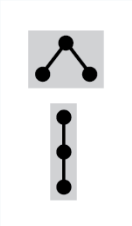
Multiset application example

Q. How many unordered trees with N nodes?

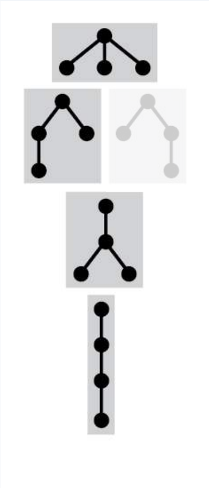
$H_1 = 1$



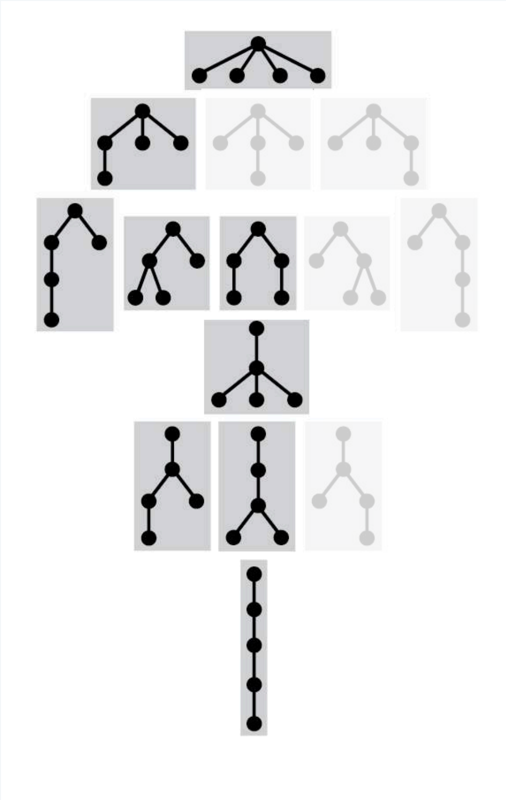
$H_2 = 1$



$H_3 = 2$



$H_4 = 4$

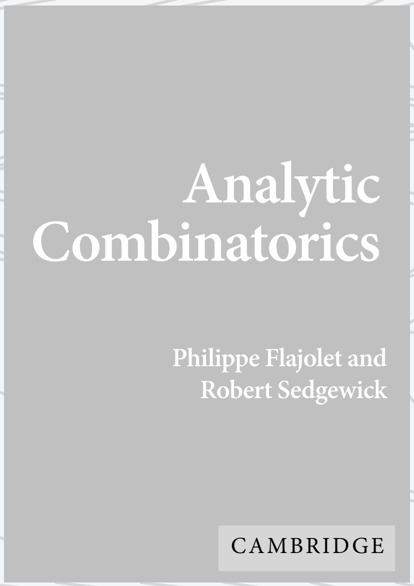


$H_5 = 9$

Combinatorial class: H, the class of all unordered trees

Construction: $H = \bullet \times MSET(H)$ ← "a tree is a node and a multiset of trees"

OGF equation: $H(z) = z \exp\big(H(z) + H(z^2)/2 + H(z^3)/3 + \ldots\big)$



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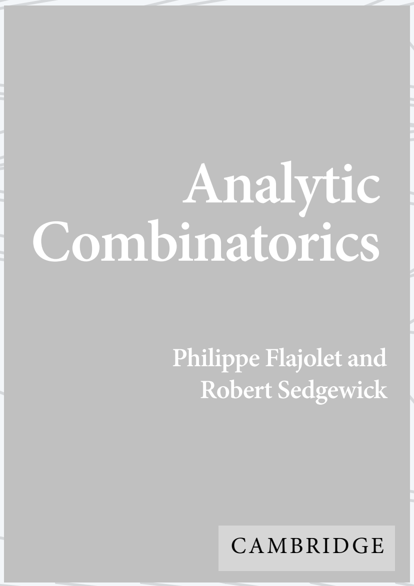
1. Combinatorial structures and OGFs

- Symbolic method
- Trees and strings
- **Powersets and multisets**
- Compositions and partitions
- Substitution



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1. Combinatorial structures and OGFs

- Symbolic method
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- **Compositions and partitions**
- Substitution

Compositions

Q. How many ways to express N as a sum of positive integers?

1	1 + 1	1 + 1 + 1	1 + 1 + 1 + 1	1 + 1 + 1 + 1 + 1
	2	1 + 2	1 + 1 + 2	1 + 1 + 1 + 2
		2 + 1	1 + 2 + 1	1 + 1 + 2 + 1
		3	1 + 3	1 + 1 + 3
			2 + 1 + 1	1 + 2 + 1 + 1
			2 + 2	1 + 2 + 2
			3 + 1	1 + 3 + 1
			4	1 + 4
				2 + 1 + 1 + 1
				2 + 1 + 2
				2 + 2 + 1
				2 + 3
				3 + 1 + 1
				3 + 2
				4 + 1
				5
$I_1 = 1$	$I_2 = 2$	$I_3 = 4$	$I_4 = 8$	$I_5 = 16$

A. $I_N = 2^{N-1}$

Integers as a combinatorial class

Atom	notation	size	GF
	$\bullet$	1	z

Combinatorial class — I, the class of all positive integers

Example — $\bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet$ ← unary notation for 7

OGF

$$I(z) = \sum_{i \in I} z^{|i|} = \sum_{N \geq 0} I_N z^N$$

Construction

$$I = SEQ_{>0}(\bullet)$$

OGF equation

$$I(z) = \frac{z}{1-z}$$

Expansion

$$I_N = 1 \text{ for } N > 0$$ ✓

Compositions

Combinatorial class C, the class of all compositions

Example ● | ●●● | ● | ●●●●● | ●● = ●●●●●●●●●●●●

← unary notation for 1 + 3 + 1 + 5 + 2 = 12

OGF

$$C(z) = \sum_{c \in C} z^{|c|} = \sum_{N \geq 0} C_N z^N$$

Construction

$$C = SEQ(I)$$

← "a composition is a sequence of positive integers"

OGF equation

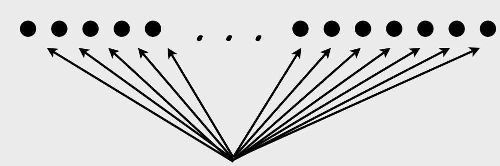
$$C(z) = \frac{1}{1 - I(z)}$$

$$= \frac{1}{1 - \frac{z}{1-z}} = \frac{1 - z}{1 - 2z}$$

Expansion

$$C_N = 2^N - 2^{N-1} = 2^{N-1} \text{ for } N > 0$$

✓



N–1 spaces between dots
each could have a bar or not
$=2^{N-1}$ possibilities ✓

Partitions

Q. How many ways to express *N* as a sum of *unordered* positive integers?

1

$P_1 = 1$

1 + 1
2

$P_2 = 2$

1 + 1 + 1
1 + 2
2 + 1
3

$P_3 = 3$

1 + 1 + 1 + 1
1 + 1 + 2
1 + 2 + 1
1 + 3
2 + 1 + 1
2 + 2
3 + 1
4

$P_4 = 5$

1 + 1 + 1 + 1 + 1
1 + 1 + 1 + 2
1 + 1 + 2 + 1
1 + 1 + 3
1 + 2 + 1 + 1
1 + 2 + 2
1 + 3 + 1
1 + 4
2 + 1 + 1 + 1
2 + 1 + 2
2 + 2 + 1
2 + 3
3 + 1 + 1
3 + 2
4 + 1
5

$P_5 = 7$

representations of the same partition

keep the one whose parts are nonincreasing

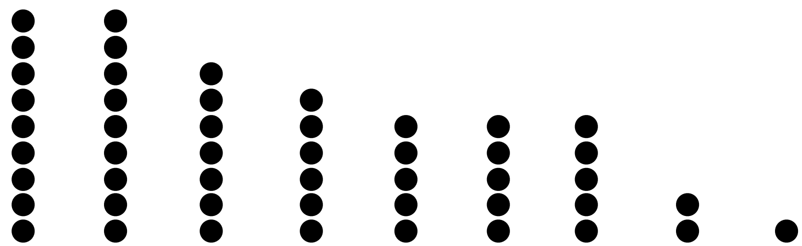
A. *Not so obvious* !

Ferrers diagrams

Def. A *Ferrers diagram* is a 2D representation of a partition: one column of dots per part.

partition $8 + 8 + 6 + 5 + 4 + 4 + 4 + 2 + 1 = 42$

Ferrers diagram



← *42 dots*

Q. How many Ferrers diagrams with *N* dots?

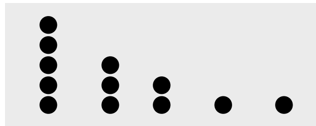
A. *Not so obvious* [need symbolic method plus saddle-point asymptotics—stay tuned]

Applications. AofA, representation theory, Lie algebras, particle physics, . . .

Partitions

Combinatorial class — P, the class of all partitions

Example



Ferrers diagram for
5 + 3 + 2 + 1 + 1 = 12

OGF

$$P(z) = \sum_{p \in P} z^{|p|} = \sum_{N \geq 0} P_N z^N$$

Construction

$$P \ = MSET\,(I\,)$$

"a partition is a *multiset* of positive integers"

OGF equation

$$P(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)\ldots}$$

$$MSET(A) \equiv \prod_{a \in A} SEQ(\{a\})$$

$$\prod_{a \in A} \frac{1}{(1-z^{|a|})} = \prod_{N \geq 0} \frac{1}{(1-z^N)^{A_N}}$$

Expansion

$$P_N \sim \frac{e^{\pi\sqrt{2N/3}}}{4N\sqrt{3}}$$

Classic result of Hardy and Ramanujan
(need saddle-point asymptotics)

Some variations on compositions and partitions

Restricted compositions

$T = \{$ any subset of $I\}$

$C^T = SEQ(SEQ_T(Z))$

$$C^T(z) = \frac{1}{1 - T(z)}$$

Compositions

$C = SEQ(I)$

$$C(z) = \frac{1 - z}{1 - 2z}$$

Compositions into M parts

$C_M = SEQ_M(I)$

$$C_M(z) = \frac{z^M}{1 - z^M}$$

Partitions into distinct parts

$Q = PSET(I)$

$$Q(z) = (1 + z)(1 + z^2)(1 + z^3)\ldots$$

Partitions

$P = MSET(I)$

$$P_N \sim \frac{e^{\pi\sqrt{2N/3}}}{4N\sqrt{3}}$$

Restricted partitions

$T = \{$ any subset of $I\}$

$P^T = MSET(SEQ_T(Z))$

$$P^T(z) = \prod_{N \in T} \frac{1}{1 - z^N}$$

In-class exercises

Q. OGF for compositions into parts less than or equal to *R* ?

Q. How many partitions into parts that are powers of 2?

A. **1**

$$\prod_{j\geq 0}(1+z^{2^j}) = (1+z)(1+z^2)(1+z^4)(1+z^8)\ldots$$
$$= (1+z+z^2+z^3)(1+z^4)(1+z^8)\ldots$$
$$= (1+z+z^2+z^3+z^4+z^5+z^6+z^7)(1+z^8)\ldots$$
$$= 1+z+z^2+z^3+z^4+z^5+z^6+z^7+z^8+z^9+z^{10}+\ldots$$

Q. How many ways to represent an integer as a sum of powers of 2?

A. **1**

$$\prod_{j\geq 0}(1+z^{2^j}) = \frac{1}{1-z}$$

How many ways to change a dollar?

Q. How many ways to change a dollar with quarters ?

A. **1**

$$[z^{100}]\frac{1}{1-z^{25}} = [z^{100}](1+z^{25}+z^{50}+\ldots) = 1$$

Q. How many ways to change a dollar with quarters *and dimes*?

A. **3**

$$[z^{100}]\frac{1}{1-z^{25}}\frac{1}{1-z^{10}} = [z^{100}](1+z^{25}+z^{50}+\ldots)(1+z^{10}+z^{20}+\ldots)$$

$$= [z^{100}](1+z^{50}+z^{100})(1+z^{50}+z^{100})$$

How many ways to change a dollar?

Q. How many ways to change a dollar with quarters ?

A. **1** $$[z^{100}]\frac{1}{1-z^{25}} = [z^{100}](1+z^{25}+z^{50}+\ldots) = 1$$

Q. How many ways to change a dollar with quarters *and dimes* ?

A. **3** $$[z^{100}]\frac{1}{1-z^{25}}\frac{1}{1-z^{10}} = [z^{100}](1+z^{25}+z^{50}+\ldots)(1+z^{10}+z^{20}+\ldots)$$

Q. How many ways to change a dollar with quarters, dimes *and nickels* ?

A. ? $$[z^{100}]\frac{1}{1-z^{25}}\frac{1}{1-z^{10}}\frac{1}{1-z^{5}}$$ ← need a computer?

Q. How many ways to change a dollar with quarters, dimes, nickels *and pennies* ?

A. ? $$[z^{100}]\frac{1}{1-z^{25}}\frac{1}{1-z^{10}}\frac{1}{1-z^{5}}\frac{1}{1-z}$$ ← need a computer?

How many ways to change a dollar?

Key insight (Pólya): If $b(z) = a(z)\dfrac{1}{1 - z^M}$ then $b(z)(1 - z^M) = a(z)$ and therefore $b_n = b_{n-M} + a_n$

Gives an easy way to compute small values by hand.

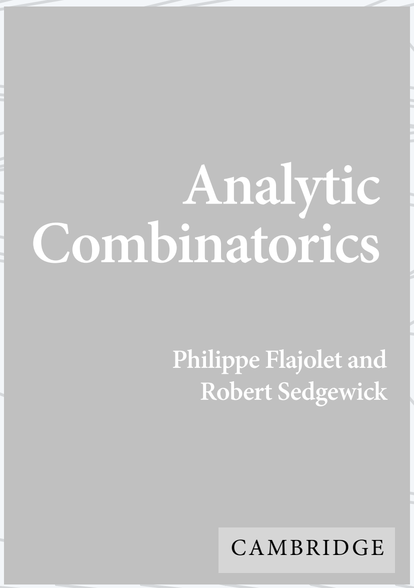
	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
$[z^n]\dfrac{1}{1-z}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$[z^n]\dfrac{1}{1-z}\dfrac{1}{1-z^5}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$[z^n]\dfrac{1}{1-z}\dfrac{1}{1-z^5}\dfrac{1}{1-z^{10}}$	1	2	4	6	9	12	16	20	25	30	36	42	49	56	64	72	81	90	100	110	121
$[z^n]\dfrac{1}{1-z}\dfrac{1}{1-z^5}\dfrac{1}{1-z^{10}}\dfrac{1}{1-z^{25}}$	1					13					49					121					242

In-class exercise

For whatever reason, the government switches to 20-cent pieces instead of dimes.

How many ways to change a dollar?

	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
$[z^n]\frac{1}{1-z}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$[z^n]\frac{1}{1-z}\frac{1}{1-z^5}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$[z^n]\frac{1}{1-z}\frac{1}{1-z^5}\frac{1}{1-z^{10}}$	1	2	3	4	6	8	10	12	15	18	21	24	28	32	36	40	45	50	55	60	66
$[z^n]\frac{1}{1-z}\frac{1}{1-z^5}\frac{1}{1-z^{10}}\frac{1}{1-z^{25}}$	1					9					30					70					136



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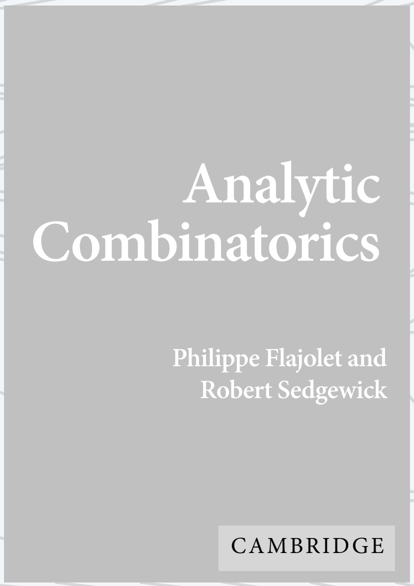
1. Combinatorial structures and OGFs

- Symbolic method
- Trees and strings
- Powersets and multisets
- **Compositions and partitions**
- Substitution



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- **Substitution**

The symbolic method for unlabeled objects (summary)

operation	notation	semantics	OGF
disjoint union	$A + B$	disjoint copies of objects from A and B	$A(z) + B(z)$
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from A and one from B	$A(z)B(z)$
sequence	$SEQ(A)$	sequences of objects from A	$\dfrac{1}{1 - A(z)}$
powerset	$PSET(A)$	finite sets of objects from A (no repetitions)	$\prod_{n\geq 1}(1 + z^n)^{A_n} = \exp\Big(-\sum_{k\geq 1}\dfrac{(-1)^k A(z^k)}{k}\Big)$
multiset	$MSET(A)$	finite sets of objects from A (with repetitions)	$\prod_{n\geq 1}\dfrac{1}{(1 - z^n)^{A_n}} = \exp\Big(\sum_{k\geq 1}\dfrac{A(z^k)}{k}\Big)$

Additional constructs are available (and still being invented)—one example to follow

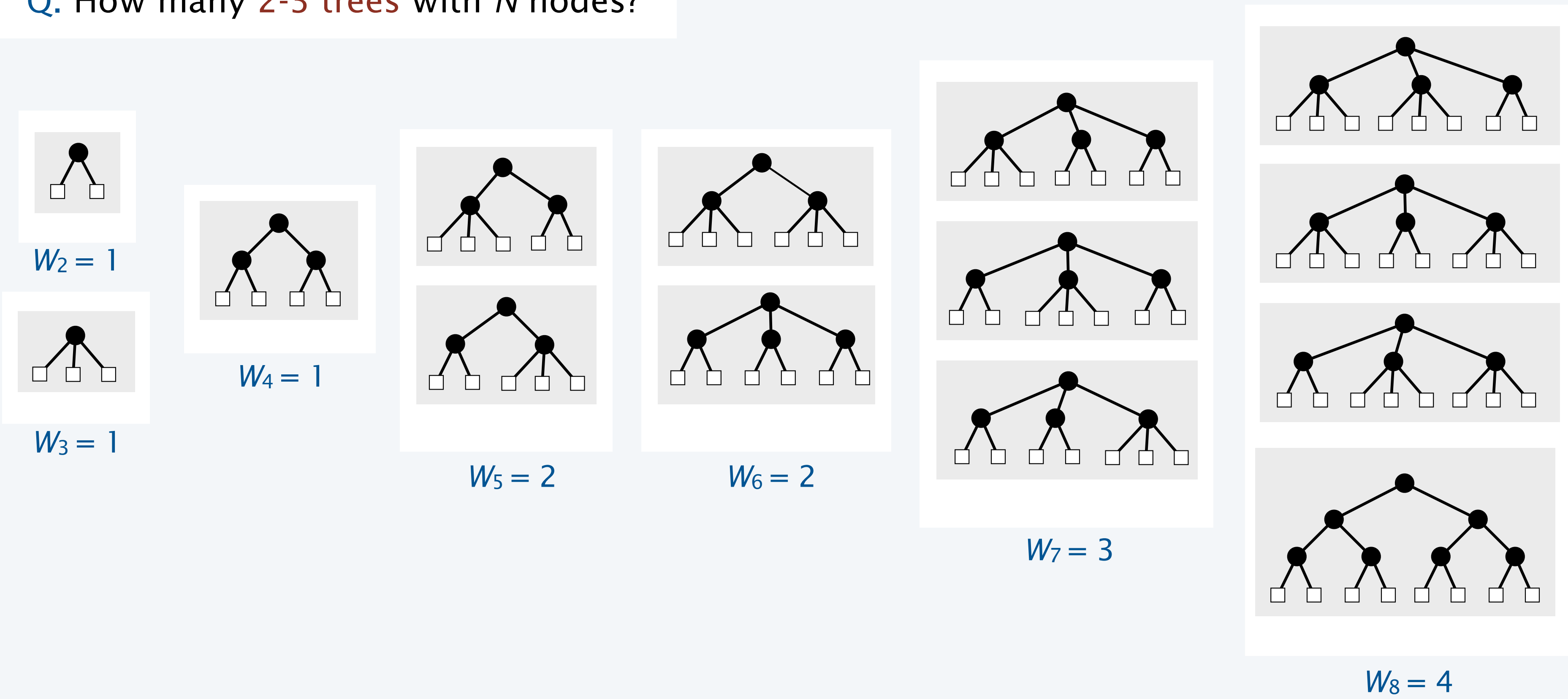
Another construct for the symbolic method: substitution

Suppose that *A* and *B* are classes of unlabeled objects with enumerating OGFs $A(z)$ and $B(z)$.

operation	notation	semantics	OGF
substitution	$A \circ [\, B \,]$	replace each object in an instance of A with an object from B	$A(B(z))$

Substitution application example

Q. How many 2-3 trees with N nodes?



Substitution application example

Q. How many 2-3 trees with N nodes?

Combinatorial class	W, the class of all 2-3 trees
Construction	$W = Z + W \circ [\,(Z \times Z) + (Z \times Z \times Z)\,]$
OGF equation	$W(z) = z + W(z^2 + z^3)$

"a 2-3 tree is a 2-3 tree with each external node replaced by a 2-node or a 3-node"

$$W(z) = z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 3z^7 + 4z^8 + \ldots$$

$$W(z^2 + z^3) = z^2 + z^3 + (z^2 + z^3)^2 + (z^2 + z^3)^3 + (z^2 + z^3)^4 + \ldots$$

$$= z^2 + z^3 + (z^4 + 2z^5 + z^6) + (z^6 + 3z^7 + 3z^8 + z^9) + z^8 + \ldots \checkmark$$

Coefficient asymptotics are complicated (oscillations in the leading term).

See A. Odlyzko, *Periodic oscillations of coefficients of power series that satisfy functional equations*, Adv. in Mathematics (1982).

Two French mathematicians on the utility of GFs

"A property... is understood better, when one constructs a bijection... than when one calculates the coefficients of a polynomial whose variables have no particular meaning. The method of generating functions, which has had devastating effects for a century, has fallen into obsolescence, for this reason.

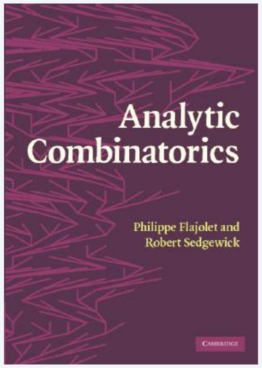
— Claude Bergé, 1968

" Generating functions are the central objects of the theory, rather than a mere artifact to solve recurrences, as it is still often believed. "

— Philippe Flajolet, 2007

Analytic combinatorics overview

To analyze properties of a large combinatorial structure:



1. Use the symbolic method
   - Define a *class* of combinatorial objects.
   - Define a notion of *size* (and associated generating function)
   - Use standard operations to develop a *specification* of the structure.

Result: A direct derivation of a GF equation (implicit or explicit).

Important note: GF equations vary widely in nature

$$P(z) = \frac{1}{(1-z)(1-z^2)(1-z^3)\ldots}$$

$$C(z) = \frac{1}{1-I(z)}$$

$$T(z) = z + T(z^2 + z^3)$$

$$B(z) = \frac{1}{1-2z}$$

$$S_M(z) = \frac{1}{(1-z)^M}$$

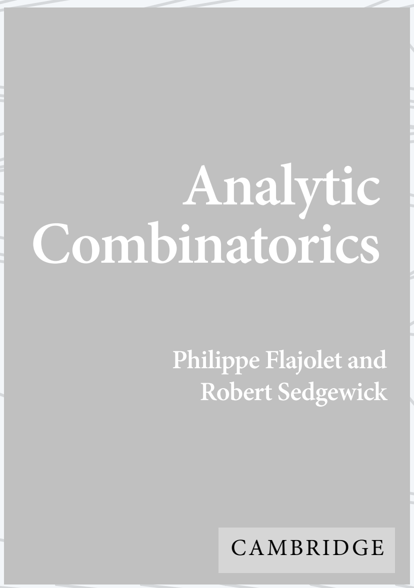
$$H(z) = z\exp\big(H(z) + H(z^2)/2 + H(z^3/3 + \ldots\big)$$

$$B_P(z) = \frac{1-z^P}{1-2z+z^{P+1}}$$

$$G(z)^2 - G(z) + z = 0$$

$$Q(z) = (1+z)(1+z^2)(1+z^3)\ldots$$

2. Use complex asymptotics to estimate growth of coefficients (stay tuned).



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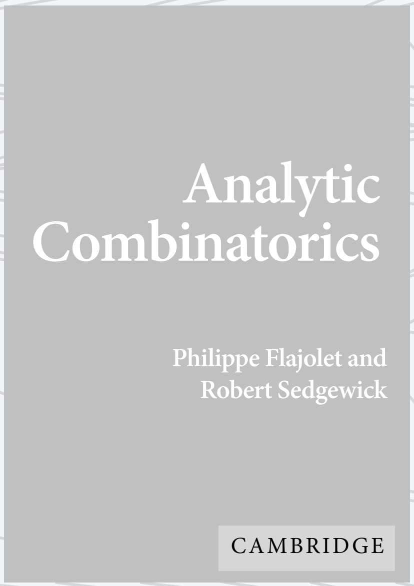
1. Combinatorial structures and OGFs

- Symbolic method
- Trees and strings
- Powersets and multisets
- Compositions and partitions
- **Substitution**



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1. Combinatorial structures and OGFs

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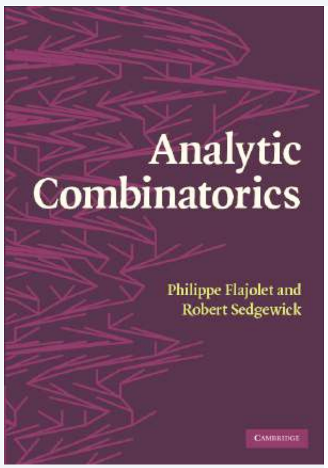
- Substitution
- **Exercises**

Note 1.23

Alice, Bob, and coding bounds



▷ **I.23.** *Alice, Bob, and coding bounds.* Alice wants to communicate n bits of information to Bob over a channel (a wire, an optic fibre) that transmits 0,1-bits but is such that any occurrence of 11 terminates the transmission. Thus, she can only send on the channel an encoded version of her message (where the code is of some length $\ell \geq n$) that does not contain the pattern 11.

Here is a first coding scheme: given the message $m = m_1 m_2 \cdots m_n$, where $m_j \in \{0, 1\}$, apply the substitution: $0 \mapsto 00$ and $1 \mapsto 10$; terminate the transmission by sending 11. This scheme has $\ell = 2n + O(1)$, and we say that its *rate* is 2. Can one design codes with better rates? with rates arbitrarily close to 1, asymptotically?

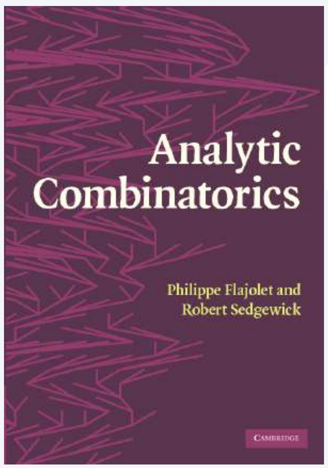
Let $\mathcal{C}$ be the class of allowed code words. For words of length n, a code of length $L \equiv L(n)$ is achievable only if there exists a one-to-one mapping from $\{0, 1\}^n$ into $\bigcup_{j=0}^{L} \mathcal{C}_j$, i.e., $2^n \leq \sum_{j=0}^{L} C_j$. Working out the OGF of $\mathcal{C}$, one finds that necessarily

$$L(n) \geq \lambda n + O(1), \qquad \lambda = \frac{1}{\log_2 \varphi} \doteq 1.440420, \quad \varphi = \frac{1+\sqrt{5}}{2}.$$

Thus no code can achieve a rate better than 1.44; i.e., a loss of at least 44% is unavoidable.

Note 1.43

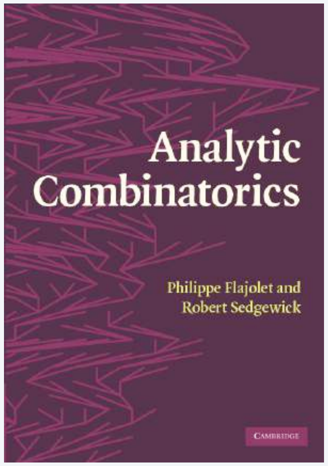
Calculating Cayley numbers and partition numbers



▷ **I.43.** *Fast determination of the Cayley–Pólya numbers.* Logarithmic differentiation of $H(z)$ provides for the H_n a recurrence by which one computes H_n in time polynomial in n. (Note: a similar technique applies to the partition numbers P_n; see p. 42.) ◁

Assignments

1. Read pages 15-94 in text.

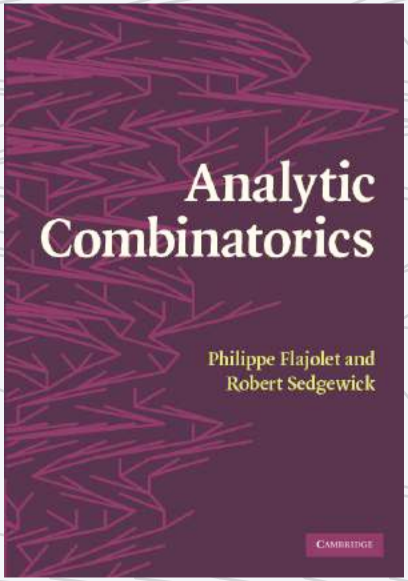


2. Write up solutions to Notes 1.23 and 1.43.

3. Programming exercises.

Program I.1. Determine the choice of four coins that maximizes the number of ways to change a dollar.

Program I.2. Write programs that estimate the rate of growth of the Cayley numbers and the partition numbers (H_n/H_{n-1} and P_n/P_{n-1}). See Note I.43.



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1. Combinatorial structures and OGFs