5. Applications of Rational and Meromorphic Asymptotics
Analytic combinatorics overview

A. SYMBOLIC METHOD
1. OGFs
2. EGFs
3. MGFs

B. COMPLEX ASYMPTOTICS
4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point
Analytic transfer for meromorphic GFs: \( f(z)/g(z) \sim c \beta^N \)

- Compute the dominant pole \( \alpha \) (smallest real with \( g(z) = 0 \)).
- Compute the residue \( h_{-1} = -f(\alpha)/g'(\alpha) \).
- Constant \( c \) is \( h_{-1}/\alpha \).
- Exponential growth factor \( \beta \) is \( 1/\alpha \)

Bottom line from last lecture

This lecture: Numerous applications
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- Compositions
- Supercritical sequence schema
Warmup: Bitstrings

How many bitstrings of length $N$?

$B_0 = 1$

$B_1 = 2$

$B_2 = 4$

$B_3 = 8$

$B_4 = 16$

Counting sequence

$\begin{array}{c}
0000 \\
0001 \\
0010 \\
0011 \\
0100 \\
0101 \\
0110 \\
0111 \\
1000 \\
1001 \\
1010 \\
1011 \\
1100 \\
1101 \\
1110 \\
1111 \\
\end{array}$

$\begin{array}{c}
000 \\
001 \\
010 \\
011 \\
100 \\
101 \\
110 \\
111 \\
\end{array}$

$B_N = 2^N$

$OGF$

$\frac{1}{1 - 2z}$

$\sum_{N \geq 0} 2^N z^N = \sum_{N \geq 0} (2z)^N = \frac{1}{1 - 2z}$
**Warmup: Bitstrings**

**Specication**

Symbolic transfer

**GF equation**

Analytic transfer

**Asymptotics**

\[ B, \text{ the class of all bitstrings} \]

\[ B = E + (Z_0 + Z_1) \times B \]

\[ B(z) = \frac{1}{1 - 2z} \]

Dominant singularity: *pole* at \( \alpha = 1/2 \)

Residue:

\[ h_{-1} = -\frac{f(z)}{g'(z)} = \frac{1}{2} \]

Coefficient of \( z^N \):

\[ \sim \frac{h_{-1}}{\alpha} \left( \frac{1}{\alpha} \right)^N = 2^N \]
Example 1: Bitstrings with restrictions on consecutive 0s

How many bitstrings of length \( N \) have no two consecutive 0s?
Example 1: Bitstrings with restrictions on consecutive 0s

\( B_{00}, \) the class of all bitstrings having no 00

\[ B_{00} = E + Z_0 + (Z_0 + Z_0 \times Z_1) \times B_{00} \]

\[ B_{00}(z) = \frac{1 + z}{1 - z - z^2} \]

\[ [z^N]B_{00}(z) = \frac{\phi^2}{\sqrt{5}} \phi^N \]

\[ \sim c_2 \beta_2^N \quad \text{with} \quad \begin{cases} \beta_2 \doteq 1.61803 \\ c_2 \doteq 1.17082 \end{cases} \]

\[ \hat{\phi} = \frac{\sqrt{5} - 1}{2} \]

\[ \phi = \frac{\sqrt{5} + 1}{2} \]

Dominant singularity: pole at \( \hat{\phi} \)

Residue: \( h_{-1} = -\frac{f'(\hat{\phi})}{g'(\hat{\phi})} = \frac{1 + \hat{\phi}}{1 + 2\hat{\phi}} \)

Coefficient of \( z^N \): \( \sim \frac{h_{-1} \left( \frac{1}{\phi} \right)^N}{\phi} = \frac{1 + \hat{\phi}}{\phi + 2\hat{\phi}^2} \phi^N \)

\[ \phi \hat{\phi} = 1 \]

\[ \phi^2 = \phi + 1 \]
Example 1: Bitstrings with restrictions on consecutive 0s

**Specification**

$B_4$, the class of all bitstrings having no $0^4$

$$B_4 = Z_{<4}(E + Z_1 B_4)$$

$$B_4(z) = (1 + z + z^2 + z^3)(1 + zB_4(z))$$

$$= \frac{1 + z + z^2 + z^3}{1 - z - z^2 - z^3 - z^4}$$

**GF equation**

Dominant singularity: *pole* at $\alpha$

Residue: $h_{-1} = -\frac{f(z)}{g'(z)} = \frac{1 + \alpha + \alpha^2 + \alpha^3}{\alpha + 2\alpha + 3\alpha^2 + 4\alpha^3}$

**Analytic transfer**

Coefficient of $z^N$:

$$[z^N]B_4(z) \sim \frac{h_{-1}}{\alpha} \left( \frac{1}{\alpha} \right)^N$$

**Asymptotics**

$$[z^N]B_4(z) \sim c_4 \beta_4^N$$

with

$$\left\{ \begin{array}{l}
\beta_4 \doteq 1.9276 \\
c_4 \doteq 1.0917
\end{array} \right.$$
Example 1: Bitstrings with restrictions on consecutive 0s

\[
\frac{1 + z}{1 - z - z^2}
\]

\[
\frac{1 + z + z^2}{1 - z - z^2 - z^3}
\]

\[
\frac{1 + z + z^2 + z^3}{1 - z - z^2 - z^3 - z^4}
\]

\[
\frac{1 + z + z^2 + z^3 + z^4}{1 - z - z^2 - z^3 - z^4 - z^5}
\]
Example 1: Bitstrings with restrictions on consecutive 0s

\[
\frac{1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9}{1 - z - z^2 - z^3 - z^4 - z^5 - z^6 - z^7 - z^8 - z^9 - z^{10}}
\]
Information on consecutive 0s in GFs for strings

(From AC Part I Lecture 5)

\[ B_M(z) = \sum_{b \in B_M} z^{|b|} = \sum_{N \geq 0} \{\# \text{ of bitstrings of length } N \text{ with no } 0^M\} z^N \]
\[ = \frac{1 + z + z^2 + \ldots + z^{M-1}}{1 - z - z^2 - \ldots z^M} = \frac{1 - z^M}{1 - 2z + z^{M+1}} \]

\[ B_M(z/2) = \sum_{N \geq 0} (\{\# \text{ of bitstrings of length } N \text{ with no runs of } M \text{ 0s}\}/2^N) z^N \]

\[ B_M(1/2) = \sum_{N \geq 0} \{\# \text{ of bitstrings of length } N \text{ with no runs of } M \text{ 0s}\}/2^N \]
\[ = \sum_{N \geq 0} \Pr \{1st N \text{ bits of a random bitstring have no runs of } M \text{ 0s}\} \]
\[ = \sum_{N \geq 0} \Pr \{\text{position of end of first } 0^M \text{ is } > N \} = \text{Expected position of end of first } 0^M \]

**Theorem.** Probability that an \( N \)-bit random bitstring has no \( 0^M \): \([z^N]B_M(z/2) \sim c_M(\beta_M/2)^N\)

**Theorem.** Expected wait time for the first \( 0^M \) in a random bitstring: \( B_M(1/2) = 2^{M+1} - 2 \)
The probability that an \(N\)-bit random bitstring does not contain 0000 is \(\sim 1.0917 \times 0.96328^N\).

The expected wait time for the first occurrence of 0000 in a random bitstring is 30.

Q. Do the same results hold for 0001?
A. NO!

Q. What is the probability that an \(N\)-bit random bitstring does not contain 0001?

Q. What is the expected wait time for the first occurrence of 0001 in a random bitstring?

Observation. Consider first occurrence of 000.
- 0000 and 0001 equally likely, BUT
- mismatch for 0000 means 0001, so need to wait four more bits
- mismatch for 0001 means 0000, so next bit could give a match.
Constructions for strings without specified patterns

Cast of characters:

- $p$ — a pattern
- $S_p$ — binary strings that do not contain $p$
- $T_p$ — binary strings that end in $p$ and have no other occurrence of $p$

First construction

- $S_p$ and $T_p$ are disjoint
- the empty string is in $S_p$
- adding a bit to a string in $S_p$ gives a string in $S_p$ or $T_p$

$$S_p + T_p = E + S_p \times \{Z_0 + Z_1\}$$

$p$ 101001010

$S_p$ 10111110101101001100110000011111

$T_p$ 1011111010110100110011001101001010
Every pattern has an autocorrelation polynomial

- slide the pattern to the left over itself.
- for each match of $i$ trailing bits with the leading bits include a term $z^{|p| - i}$

```
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
101001010
```

$$c_{101001010}(z) = 1 + z^5 + z^7$$
Constructions for bitstrings without specified patterns

Second construction
• for each 1 bit in the autocorrelation of any string in \( T_p \) add a “tail”
• result is a string in \( S_p \) followed by the pattern

\[
p = 101001010
\]

\[
a \text{ string in } T_p = 1011110101101001100110101001010
\]

\[
\text{strings in } S_p = 1011110101101001100110101001010
\]

\[
S_p \times \{p\} = T_p \times \sum_{c_i \neq 0} \{t_i\}
\]
Bitstrings without specified patterns

How many \( N \)-bit strings do not contain a specified pattern \( p \) ?

<table>
<thead>
<tr>
<th>Classes</th>
<th>( S_p ) — the class of binary strings with no ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( T_p ) — the class of binary strings that end in ( p ) and have no other occurrence</td>
</tr>
</tbody>
</table>

| OGFs    | \( S_p(z) = \sum_{s \in S_p} z^{|s|} \) |
|---------|-------------------------------------|
|         | \( T_p(z) = \sum_{s \in T_p} z^{|s|} \) |

Constructions

\[
S_p + T_p = E + S_p \times \{Z_0 + Z_1\} \quad S_p \times \{p\} = T_p \times \sum_{c_i \neq 0} \{t_i\}
\]

OGF equations

\[
S_p(z) + T_p(z) = 1 + 2zS_p(z) \quad S_p(z)z^p = T_p(z)c_p(z)
\]

Solution

\[
S_p(z) = \frac{c_p(z)}{z^p + (1 - 2z)c_p(z)}
\]

Extract coefficients

\[
[z^N]S_p(z) \sim c_p \beta_p^N \quad \text{where } \begin{cases} 
\beta_p \text{ is the dominant root of } z^p + (1 - 2z)c_p(z) \\
c_p = \text{ [explicit formula available]} 
\end{cases}
\]
Bitstrings without specified patterns

- Specification
  - Symbolic transfer
  - GF equation
  - Analytic transfer
  - Asymptotics
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Example 2: Derangements

How many permutations of size $N$ have no singleton cycles?
**Example 2: Derangements**

**Specification**

- **D**, the class of all permutations with no singleton cycles

\[
D = \text{SET}(\text{CYC}_{>1}(Z))
\]

**GF equation**

\[
D(z) = \frac{e^{-z}}{1 - z}
\]

**Analytic transfer**

- Dominant singularity: pole at 1
- Residue: \( h_{-1} = -\frac{f(1)}{g'(1)} = e^{-1} \)

\[
[z^N]D(z) = \frac{h_{-1}}{1}1^N = \frac{1}{e}
\]

**Asymptotics**

\[
N! [z^N] D(z) \sim \frac{N!}{e}
\]

**Estimates**

- Estimates are extremely accurate even for small \( N \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( N!e )</th>
<th>( D_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.7357...</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2.2072...</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>8.8291...</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>44.1455...</td>
<td>44</td>
</tr>
</tbody>
</table>
Example 2: Derangements

\( D_M \), the class of all permutations with no cycles of length \( \leq M \)

\[
D_M = \text{SET}(\text{CYC}_{>M}(Z))
\]

\[
D_M(z) = \frac{e^{-z} - z^2}{2} - \frac{z^3}{3} - \ldots - \frac{z^M}{M} \quad \frac{1}{1 - z}
\]

Dominant singularity: pole at 1

Residue: \( h_{-1} = -\frac{f(1)}{g'(1)} = e^{-H_M} \)

\[
[z^N]D(z) = \frac{h_{-1}}{1}1^N = \frac{1}{e^{H_M}}
\]

Analytic transfer

Asymptotics

GF equation

Symbolic transfer

Specification
Example 2: Derangements

\[ \frac{e^{-z}}{1 - z} \]

\[ \frac{e^{-z - z^2/2}}{1 - z} \]

\[ \frac{e^{-z - z^2/2 - z^3/3}}{1 - z} \]

\[ \frac{e^{-z - z^2/2 - z^3/3 - z^4/4}}{1 - z} \]
Example 2: Derangements

\[ \frac{e^{-z^2/2 - z^3/3 - z^4/4 - z^5/5 - z^6/6 - z^7/7 - z^8/8 - z^9/9 - z^{10}/10}}{1 - z} \]
Example 3: Surjections

How many words of length $N$ are $M$-surjections for some $M$?

$R_1 = 1$

$R_2 = 3$

$R_3 = 13$

$R_4 = 75$

"coupon collector sequences"

For some $M$, each of the first $M$ letters appears at least once.
Example 3: Surjections

\[ R, \text{ the class of all surjections} \]

\[ R = \text{SEQ}(\text{SET}_{>0}(\mathbb{Z})) \]

\[ R(z) = \frac{1}{1 - (e^z - 1)} \]

\[ = \frac{1}{2 - e^z} \]

\[ [z^N]R(z) = \frac{1}{2(\ln 2)^{N+1}} \]

**Dominant singularity:** pole at \( z = \ln 2 \)

**Residue:** \( h_{-1} = -\frac{1}{g'(\ln 2)} = \frac{1}{2} \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \frac{N}{2}(\ln 2)^{N+1} )</th>
<th>( R_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.0027...</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>12.9962...</td>
<td>13</td>
</tr>
<tr>
<td>4</td>
<td>74.9987...</td>
<td>75</td>
</tr>
</tbody>
</table>

estimates are extremely accurate even for small \( N \)
Example 3: Surjections

How many words of length $N$ are double surjections for some $M$?

\[ R_2 = 1 \]
\[ R_3 = 1 \]
\[ R_4 = 7 \]
\[ R_5 = 21 \]

"double coupon collector sequences"

For some $M$, each of the first $M$ letters appears at least twice.
Example 3: Surjections

**Specification**

**R**, the class of all double surjections

\[ R = \text{SEQ}(\text{SET}_{>1}(Z)) \]

\[ R(z) = \frac{1}{1 - (e^z - z - 1)} = \frac{1}{2 + z - e^z} \]

\[ R_N \sim \frac{1}{\rho + 1} \frac{N!}{\rho^{N+1}} \]

**Symbolic transfer**

\[ e^z = z + 2 \]

**GF equation**

Singularities where \( e^z = z + 2 \)

Dominant singularity: **pole** at \( \rho \approx 1.14619 \)

**Analytic transfer**

Residue: \( h_{-1} = -\frac{1}{g'(\rho)} = \frac{1}{e^\rho - 1} = \frac{1}{\rho + 1} \)

**Asymptotics**

**Residue**

- \( h_{-1} = -\frac{1}{g'(\rho)} = \frac{1}{e^\rho - 1} = \frac{1}{\rho + 1} \)

**Dominant singularity:** pole at \( \rho \approx 1.14619 \)
Example 3: Surjections

\[
\frac{1}{2 - e^z}
\]

\[
\frac{1}{2 + z - e^z}
\]

\[
\frac{1}{2 + z + z^2/2 - e^z}
\]

\[
\frac{1}{2 + z + z^2/2 + z^3/6 - e^z}
\]
Example 3: Surjections

\[ \frac{1}{2 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \frac{z^8}{8!} + \frac{z^9}{9!}} - e^z \]
Example 4: Alignments

How many *sequences of labelled cycles* of size $N$?

$O_1 = 1$

$O_2 = 3$

$O_3 = 14$

$O_4 = 88$
**Example 3: Alignments**

\( \mathcal{O} \), the class of all alignments

\[ \mathcal{O} = \text{SEQ} (\text{CYC}(\mathcal{Z})) \]

\[ O(z) = \frac{\frac{1}{1 - \ln 1}}{1 - \ln \frac{1}{1 - z}} \]

**Singularities where** \( \ln \frac{1}{1 - z} = 1 \)**

**Dominant singularity: pole at** \( z = 1 - \frac{1}{e} \)

**Residue:** \( h_{-1} = -\frac{1}{g'(1 - 1/e)} = \frac{1}{e} \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( N!/(1 - 1/e)^{N+1} )</th>
<th>( O_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.9129...</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>13.8247...</td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td>87.4816...</td>
<td>88</td>
</tr>
</tbody>
</table>

Estimates are extremely accurate even for small \( N \)
### Example 4: Set partitions

Q. How many ways to partition an $N$-element set into $r$ subsets?

- $S_{N2} = 2^N - 1$
- $S_{33} = 6$
- $S_{53} = 25$

**Application: rhyming schemes**

*There was a small boy of Quebec
Who was buried in snow to his neck
When they said, "Are you friz?"
He replied, "Yes, I is —
But we don't call this cold in Quebec!*

*TWO roads diverged in a yellow wood,
And sorry I could not travel both
And be one traveler, long I stood
And looked down one as far as I could
To where it bent in the undergrowth;*
Example 4: Set partitions

\( S_r \), the class of all poems with \( r \) rhymes

\[
S_r = Z_A \times SEQ(Z_A) \times Z_B \times SEQ(Z_A + Z_B) \times Z_C \times SEQ(Z_A + Z_B + Z_C) \times ...
\]

\[
S_r(z) = \frac{z^r}{(1 - z)(1 - 2z) \ldots (1 - rz)}
\]

\[
[Z^N]S_r(z) \sim \frac{r^N}{r!}
\]

Singularities at 1, 1/2, 1/3, ..., 1/r

Dominant singularity: pole at 1/r

Residue: \( h_{-1} = -\frac{f(1/r)}{g'(1/r)} = \frac{1}{r \cdot r!} \)
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Example 5: Compositions

Q. How many ways to express $N$ as a sum of positive integers?

A. $I_N = 2^{N-1}$

$\begin{align*}
I_1 &= 1 \\
I_2 &= 2 \\
I_3 &= 4 \\
I_4 &= 8 \\
I_5 &= 16
\end{align*}$
**Example 5: Compositions**

$I$, the class of all positive integers

\[ I = \text{SEQ}_{\geq 0}(Z) \]

\[ I(z) = \frac{z}{1 - z} \]

\[ I_N = 1 \text{ for } N > 0 \]

**Singularity:** pole at 1

**Residue:** \[ h_{-1} = -\frac{f(1)}{g'(1)} = 1 \]
Example 5: Compositions

\[ C, \text{ the class of all compositions} \]

\[ C = \text{SEQ}(I) \]

\[
C(z) = \frac{1}{1 - I(z)} \\
= \frac{1}{1 - \frac{z}{1 - z}} = \frac{1 - z}{1 - 2z}
\]

\[ C_N = 2^{N-1} \text{ for } N > 0 \]

**Singularity:** pole at 1/2

**Residue:** \( h_{-1} = -\frac{f(1/2)}{g'(1/2)} = 1/4 \)
Example 5: Compositions

Q. How many ways to express $N$ as a sum of 1s and 2s?

1
$F_1 = 1$

1 + 1
$F_2 = 2$

1 + 1 + 1
$F_3 = 3$

1 + 1 + 1 + 1
$F_4 = 5$

1 + 1 + 1 + 1 + 1
$F_5 = 8$

Q. How many ways to express $N$ as a sum of 1s and 2s?

A. *Fibonacci numbers*
Example 5: Compositions

\[ F, \text{ the class of all compositions composed of } 1s \text{ and } 2s \]

\[ F = \text{SEQ}(Z + Z^2) \]

\[ F(Z) = \frac{1}{1 - Z - Z^2} \]

\[ F_N \sim \frac{\phi^N}{\sqrt{5}} \]

\[ \frac{1}{\sqrt{5}} \approx .4472 \text{ and } \phi \approx 1.618 \]

\[ \hat{\phi} = \frac{\sqrt{5} - 1}{2} \]
\[ \phi = \frac{\sqrt{5} + 1}{2} \]

Dominant singularity: pole at \( \hat{\phi} \)

Residue: \( h_{-1} = -\frac{f(\hat{\phi})}{g'(\hat{\phi})} = \frac{1}{1 + 2\hat{\phi}} \)

Coefficient of \( z^N \): \( \sim \frac{h_{-1}}{\phi} \left(\frac{1}{\phi}\right)^{N+1} = \frac{1}{1 + 2\phi} \phi^N \)

\( \phi \hat{\phi} = 1 \)
\( \phi^2 = \phi + 1 \)
\( 1 + 2\phi = \sqrt{5} \)
Q. How many ways to express $N$ as a sum of primes?
Example 5: Compositions

\( \mathbf{P} \), the class of all compositions composed of primes

\[ \mathbf{P} = \text{SEQ}(Z^2 + Z^3 + Z^5 + Z^7 + \ldots) \]

\[ P(z) = \frac{1}{1 - z^2 - z^3 - z^5 - z^7 - z^{11} - \ldots} \]

\[ [z^N]P(z) \sim \lambda \beta^N \quad \text{with} \quad \begin{cases} \beta \doteq 1.4762 \\ \lambda \doteq 0.3037 \end{cases} \]

Note: periodic oscillations are present in the next term

Dominant singularity: pole at \( 1/\beta \doteq 0.6774 \)

Interesting calculations omitted (see text)

pp. 298–299
Example 6: Denumerants (partitions from a fixed set)

Q. How many ways to make change for $N$ cents?

\[ Q_{14} = 4 \]

\[ Q_{15} = 6 \]
Example 6: Denumerants (partitions from a fixed set)

\( Q \), the class of all partitions composed of 1s, 5s, 10s, 25s

\[
Q = \text{MSET}(Z + Z^5 + Z^{10} + Z^{25})
\]

\[
Q(z) = \frac{1}{(1 - z)(1 - z^5)(1 - z^{10})(1 - z^{25})}
\]

\[
[z^N]Q(z) \sim \frac{N^3}{1 \cdot 5 \cdot 10 \cdot 25 \cdot 3!} = \frac{N^3}{7500}
\]

Dominant singularity: pole of order 5 at 1

Residue:
\[
h_{-4} = \lim_{z \to 1} (1 - z)^4 Q(z)
\]

\[
= \frac{1}{1 \cdot 5 \cdot 10 \cdot 25}
\]

\[
\lim_{z \to 1} \frac{1 - z}{1 - z^t} = \lim_{z \to 1} \frac{1}{1 + z + z^2 + \ldots + z^{t-1}} = \frac{1}{t}
\]
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5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- Restricted compositions
- Supercritical sequence schema
**Sequence schema**

**Terminology.** A *schema* is a treatment that unifies the analysis of a family of classes.

**Definition.** A class that admits a construction of the form $F = \text{SEQ}(G)$, where $G$ is any class (labelled or unlabelled) is said to be a *sequence class*, which falls within the *sequence schema*.

**Enumeration:**

- **F = SEQ(G)**
  
  \[ F(z) = \frac{1}{1 - G(z)} \]

  - **unlabelled case:** number of structures is $f_N$
  - **labelled case:** number of structures is $N! f_N$

**Parameters:**

- mark number of $G$ components with $u$
  
  \[ F = \text{SEQ}(u \ G) \rightarrow F(z, u) = \frac{1}{1 - uG(z)} \]

- mark number of $G_k$ components with $u$
  
  \[ F = \text{SEQ}(u \ G_k + G \setminus G_k) \rightarrow F^k(z, u) = \frac{1}{1 - (G(z) + (u - 1)g_k z^k)} \]
Supercritical sequence classes

**Supercriticality**: A technical condition that enables us to unify the analysis of sequence classes.

**Definition. Supercritical sequence classes.**
A sequence class \( F = \text{SEQ}(G) \) is said to be *supercritical* if \( G(\rho) > 1 \) where \( G(z) \) is the generating function associated with \( G \) and \( \rho > 0 \) is the radius of convergence of \( G(z) \).

Example: GF for integers: \( I(z) = \frac{z}{1 - z} \)

radius of convergence: \( \rho = 1 - \epsilon \) for any \( \epsilon > 0 \)

supercriticality test: \( I(1 - \epsilon) = \frac{1}{\epsilon} - 1 > 1 \) for \( \epsilon < 1/2 \)

Therefore, the class of compositions \( C = \text{SEQ}(I) \) is supercritical.

**Note:** For simplicity, we ignore periodicities in GFs in this lecture:

**Definition. Strong aperiodicity.** A GF \( G(z) \) is said to be *strongly aperiodic* when there does not exist an integer \( d > 1 \) such that \( G(z) = h(z^d) \) for some \( h(z) \) analytic at 0.
Transfer theorem for supercritical sequence classes

Theorem. Asymptotics of supercritical sequences. If $\mathbf{F} = \text{SEQ}(\mathbf{G})$ is a strongly aperiodic supercritical sequence class, then $[z^N]F(z) \sim \frac{1}{G'(\lambda)} \frac{1}{\lambda^{N+1}}$ where $\lambda$ is the root of $G(\lambda) = 1$ in $(0, \rho)$.

Proof sketch:

• $G(z)$ increases from $G(0) = 0$ to $G(\rho)>1$, so $\lambda$ is well defined.

• At $\lambda$, $G(z)$ admits the series expansion $G(z) = 1 + G'(\lambda)(z - \lambda) + G''(\lambda)(z - \lambda)^2/2! + \cdots$

• Therefore, $F(z) = 1/(1 - G(z))$ has a simple pole at $\lambda$, and $F(z) \sim -\frac{1}{G'(\lambda)(z - \lambda)} = \frac{1}{\lambda G'(\lambda)} \frac{1}{1 - z/\lambda}$
Transfer theorem for supercritical sequence classes

**Theorem. Asymptotics of supercritical sequences.** If $\mathbf{F} = \text{SEQ}(\mathbf{G})$ is a strongly aperiodic supercritical sequence class, then $[z^N]F(z) \sim \frac{1}{G'(\lambda)} \frac{1}{\lambda^{N+1}}$ where $\lambda$ is the root of $G(\lambda) = 1$ in $(0, \rho)$.

<table>
<thead>
<tr>
<th>construction</th>
<th>$F(z)$</th>
<th>$G(z)$</th>
<th>$\lambda$</th>
<th>coefficient asymptotics</th>
</tr>
</thead>
<tbody>
<tr>
<td>surjections</td>
<td>$\mathbf{R} = \text{SEQ} (\text{SET}_{&gt;0}(\mathbf{Z}))$</td>
<td>$\frac{1}{2 - e^z}$</td>
<td>$e^z - 1$</td>
<td>$\ln 2$</td>
</tr>
<tr>
<td>alignments</td>
<td>$\mathbf{O} = \text{SEQ} (\text{CYC}(\mathbf{Z}))$</td>
<td>$\frac{1}{1 - \ln \frac{1}{1 - z}}$</td>
<td>$\ln \frac{1}{1 - z}$</td>
<td>$1 - \frac{1}{e}$</td>
</tr>
<tr>
<td>compositions</td>
<td>$\mathbf{C} = \text{SEQ}(\mathbf{I})$</td>
<td>$\frac{1}{1 - \frac{z}{1-z}}$</td>
<td>$\frac{Z}{1 - z}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
**Parts in compositions**

**Q.** How many parts in a *random composition* of size $N$?

1

1 + 1

1 + 1 + 1

1 + 1 + 1 + 1

$1 + 1 + 1 + 2$

$1 + 1 + 2 + 1$

$1 + 2 + 1$

$1 + 3$

$2 + 1 + 1$

$2 + 2$

$3 + 1$

$4$

$5$

$53$
### Components in surjections

**What is the expected value of $M$ in a *random surjection* of size $N$?**

<table>
<thead>
<tr>
<th>1 1 1</th>
<th>1 1 2</th>
<th>1 2 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 1</td>
<td>1 3 2</td>
<td>2 1 3</td>
</tr>
<tr>
<td>1 2 2</td>
<td>2 1 3</td>
<td>2 3 1</td>
</tr>
<tr>
<td>2 1 1</td>
<td>2 3 1</td>
<td>3 1 2</td>
</tr>
<tr>
<td>2 1 2</td>
<td>3 1 2</td>
<td>3 2 1</td>
</tr>
</tbody>
</table>

$(1 + 2 \cdot 2)/3 \cong 1.666$

<table>
<thead>
<tr>
<th>1 1 1 1</th>
<th>1 1 1 2</th>
<th>1 1 2 3</th>
<th>1 2 3 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 2 1</td>
<td>1 1 2 2</td>
<td>1 1 3 2</td>
<td>1 1 3 4</td>
</tr>
<tr>
<td>1 2 1 1</td>
<td>1 2 1 2</td>
<td>1 2 3 2</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>2 1 1 1</td>
<td>2 1 1 2</td>
<td>2 1 3 2</td>
<td>2 1 3 4</td>
</tr>
<tr>
<td>2 1 2 1</td>
<td>2 1 2 2</td>
<td>2 1 3 2</td>
<td>2 1 3 4</td>
</tr>
<tr>
<td>2 2 1 1</td>
<td>2 2 1 1</td>
<td>2 2 1 2</td>
<td>2 2 1 2</td>
</tr>
<tr>
<td>2 2 1 1</td>
<td>2 2 1 1</td>
<td>2 2 1 2</td>
<td>2 2 1 2</td>
</tr>
</tbody>
</table>

$(1 + 2 \cdot 6 + 3 \cdot 6)/13 \cong 2.384$

"*coupon collector sequences*"

For some $M$, each of the first $M$ letters appears at least once.

$(1 + 2 \cdot 14 + 3 \cdot 36 + 4 \cdot 24)/75 \cong 3.106$
Components in alignments

How many cycles in a random alignment of size $N$?

$$(1 + 2 \cdot 2)/3 \approx 1.666$$

$$(1 \cdot 2 + 2 \cdot 6 + 3 \cdot 6)/14 \approx 2.286$$

$$(1 \cdot 12 + 2 \cdot 16 + 3 \cdot 36 + 4 \cdot 24)/88 \approx 2.818$$
A poster child for analytic combinatorics

Such questions can be answered immediately via general transfer theorems
Corollary. *Number of components in supercritical sequence classes.* If \( F = \text{SEQ}(G) \) is a strongly aperiodic supercritical sequence class, then the expected number of \( G \)-components in a random \( F \)-component of size \( N \) is

\[
\mu_N \sim \frac{N + 1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1
\]

with variance

\[
\sigma_N^2 \sim \frac{\lambda G''(\lambda) + G'(\lambda) - G'(\lambda)^2}{\lambda^2 G'(\lambda)^3} N.
\]

Proof idea:

\[
\mu_N = \frac{1}{f_N[Z^N]} \frac{\partial}{\partial u} \frac{1}{1 - uG(z)} \Big|_{u=1} = \frac{1}{f_N[Z^N]} \frac{G(z)}{(1 - G(z))^2}
\]

[further details omitted]
**Number of components in supercritical sequence classes**

**Corollary.** *Number of components in supercritical sequence classes.* If $F = \text{SEQ}(G)$ is a strongly aperiodic supercritical sequence class, then the expected number of $G$-components in a random $F$-component of size $N$ is $\mu_N \sim \frac{N + 1}{\lambda G'(\lambda)} + \frac{G''(\lambda)}{G'(\lambda)^2} - 1$ with variance $\sigma_N^2 \sim \frac{\lambda G''(\lambda) + G'(\lambda) - G'(\lambda)^2}{\lambda^2 G'(\lambda)^3}N$.  

<table>
<thead>
<tr>
<th>construction</th>
<th>$F(z)$</th>
<th>$G(z)$</th>
<th>$\lambda$</th>
<th>expected number of components</th>
</tr>
</thead>
<tbody>
<tr>
<td>compositions</td>
<td>$C = \text{SEQ}(1)$</td>
<td>$\frac{1}{1 - \frac{z}{1-z}}$</td>
<td>$\frac{z}{1-z}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>surjections</td>
<td>$R = \text{SEQ}(\text{SET}_{&gt;0}(Z))$</td>
<td>$\frac{1}{2 - e^z}$</td>
<td>$e^z - 1$</td>
<td>$\ln 2$</td>
</tr>
<tr>
<td>alignments</td>
<td>$O = \text{SEQ}(\text{CYC}(Z))$</td>
<td>$\frac{1}{1 - \ln \frac{1}{1-z}}$</td>
<td>$\ln \frac{1}{1-z}$</td>
<td>$1 - \frac{1}{e}$</td>
</tr>
</tbody>
</table>

Same idea extends to give profile of component sizes.
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- Restricted compositions
- Supercritical sequence schema
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
- Other familiar examples
- Restricted compositions
- Supercritical sequence schema
- Summary
AC via meromorphic asymptotics: summary of classic applications

<table>
<thead>
<tr>
<th>class</th>
<th>specification</th>
<th>generating function</th>
<th>coefficient asymptotics</th>
</tr>
</thead>
<tbody>
<tr>
<td>bitstrings</td>
<td>$B = E + (Z_0 + Z_1) \times B$</td>
<td>$\frac{1}{1 - 2z}$</td>
<td>$2^N$</td>
</tr>
<tr>
<td>derangements</td>
<td>$D = \text{SET}(\text{CYC}_{&gt;0}(Z))$</td>
<td>$\frac{e^{-z}}{1 - z}$</td>
<td>$\sim \frac{N!}{e}$</td>
</tr>
<tr>
<td>surjections</td>
<td>$R = \text{SEQ}(\text{SET}_{&gt;0}(Z))$</td>
<td>$\frac{1}{2 - e^z}$</td>
<td>$\sim \frac{1}{2(\ln 2)^{N+1}}$</td>
</tr>
<tr>
<td>alignments</td>
<td>$O = \text{SEQ}(\text{CYC}(Z))$</td>
<td>$\frac{1}{1 - \ln \frac{1}{1 - z}}$</td>
<td>$\sim \frac{N!}{e(1 - 1/e)^{N+1}}$</td>
</tr>
<tr>
<td>set partitions</td>
<td>$S_r = Z \times \text{SEQ}(Z) \times Z \times \text{SEQ}(Z+Z) \times \ldots$</td>
<td>$\frac{z^r}{(1 - z) \ldots (1 - rz)}$</td>
<td>$\sim \frac{r^N}{r!}$</td>
</tr>
<tr>
<td>integers</td>
<td>$I = \text{SEQ}_{&gt;0}(Z))$</td>
<td>$\frac{z}{1 - z}$</td>
<td>$1$</td>
</tr>
<tr>
<td>compositions</td>
<td>$C = \text{SEQ}(I)$</td>
<td>$\frac{1}{1 - \frac{z}{1-z}}$</td>
<td>$2^{N-1}$</td>
</tr>
</tbody>
</table>
### AC via meromorphic asymptotics: summary of classic applications (variants)

<table>
<thead>
<tr>
<th>class</th>
<th>specification</th>
<th>generating function</th>
<th>coefficient asymptotics</th>
</tr>
</thead>
<tbody>
<tr>
<td>bitstrings with no 0000</td>
<td>$B_4 = Z_{&lt;4}(E + Z_1 B_4)$</td>
<td>$\frac{1 + z + z^2 + z^3}{1 - z - z^2 - z^3 - z^4}$</td>
<td>$1.092(1.928)^N$</td>
</tr>
<tr>
<td>generalized derangements</td>
<td>$D = \text{SET}(\text{CYC}_{&gt;M}(Z))$</td>
<td>$\frac{e^{-z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots - \frac{z^M}{M}}}{1 - z}$</td>
<td>$\frac{N!}{e^{HM}}$</td>
</tr>
<tr>
<td>double surjections</td>
<td>$R = \text{SEQ}(\text{SET}_{&gt;1}(Z))$</td>
<td>$\frac{1}{2 + z - e^z}$</td>
<td>$0.4065 \frac{N!}{(1.146)^N}$</td>
</tr>
<tr>
<td>compositions of 1s and 2s</td>
<td>$F = \text{SEQ}(Z + Z^2)$</td>
<td>$\frac{1}{1 - z - z^2}$</td>
<td>$0.4472(1.618)^N$</td>
</tr>
<tr>
<td>compositions of primes</td>
<td>$P = \text{SEQ}(Z^2 + Z^3 + Z^5 + \ldots)$</td>
<td>$\frac{1}{1 - z^2 - z^3 - z^5 - z^7 - \ldots}$</td>
<td>$0.3037(1.476)^N$</td>
</tr>
<tr>
<td>denumerants</td>
<td>$Q = \text{MSET}(Z + Z^5 + Z^{10} + Z^{25})$</td>
<td>$\frac{1}{(1 - z)(1 - z^5)(1 - z^{10})(1 - z^{25})}$</td>
<td>$\frac{N^3}{7500}$</td>
</tr>
</tbody>
</table>
"If you can specify it, you can analyze it"

1. The transfer theorem for meromorphic GFs enables immediate analysis of a variety of classes.

2. Variations are handled just as easily.

3. The **supercritical sequence schema** unifies the analysis for an entire family of classes, including analysis of parameters.

**Note:** Several other schemas have been developed (see text).

**Next:** GFs that are not meromorphic (singularities are *not* poles).
5. Applications of Rational and Meromorphic Asymptotics

• Bitstrings
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• Summary
5. Applications of Rational and Meromorphic Asymptotics

- Bitstrings
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- Supercritical sequence schema
- Exercises
Web Exercise V.1

Patterns in strings.

Web Exercise V.1. Give an asymptotic expression for the number of strings that do not contain the pattern 0000000001. Do the same for 0101010101.
Web Exercise V.2

Variants of supercritical sequence classes.

Web Exercise V.2. Give asymptotic expressions for the number of objects of size $N$ and the number of parts in a random object of size $N$ for the following classes: compositions of 1s, 2s, and 3s, triple surjections, and alignments with no singleton cycles.
Assignments

1. Read pages 289-300 (*Applications of R&M Asymptotics*) in text. Skim pages 301-375. Usual caveat: Try to get a feeling for what's there, not understand every detail.

2. Write up solutions to Web exercises V.1 and V.2.


**Program V.1.** In the style of the plots in the lectures slides, plot the GFs for the set of bitstrings having no occurrence of the pattern 000000000. Do the same for 0101010101. (See Web Exercise V.1).
5. Applications of Rational and Meromorphic Asymptotics