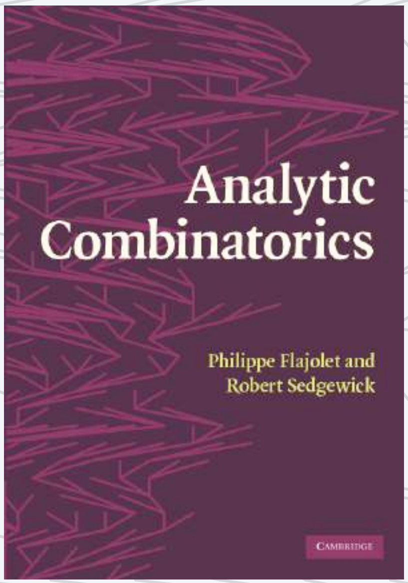
ANALYTIC COMBINATORICS

PART TWO



http://ac.cs.princeton.edu

6. Singularity Analysis

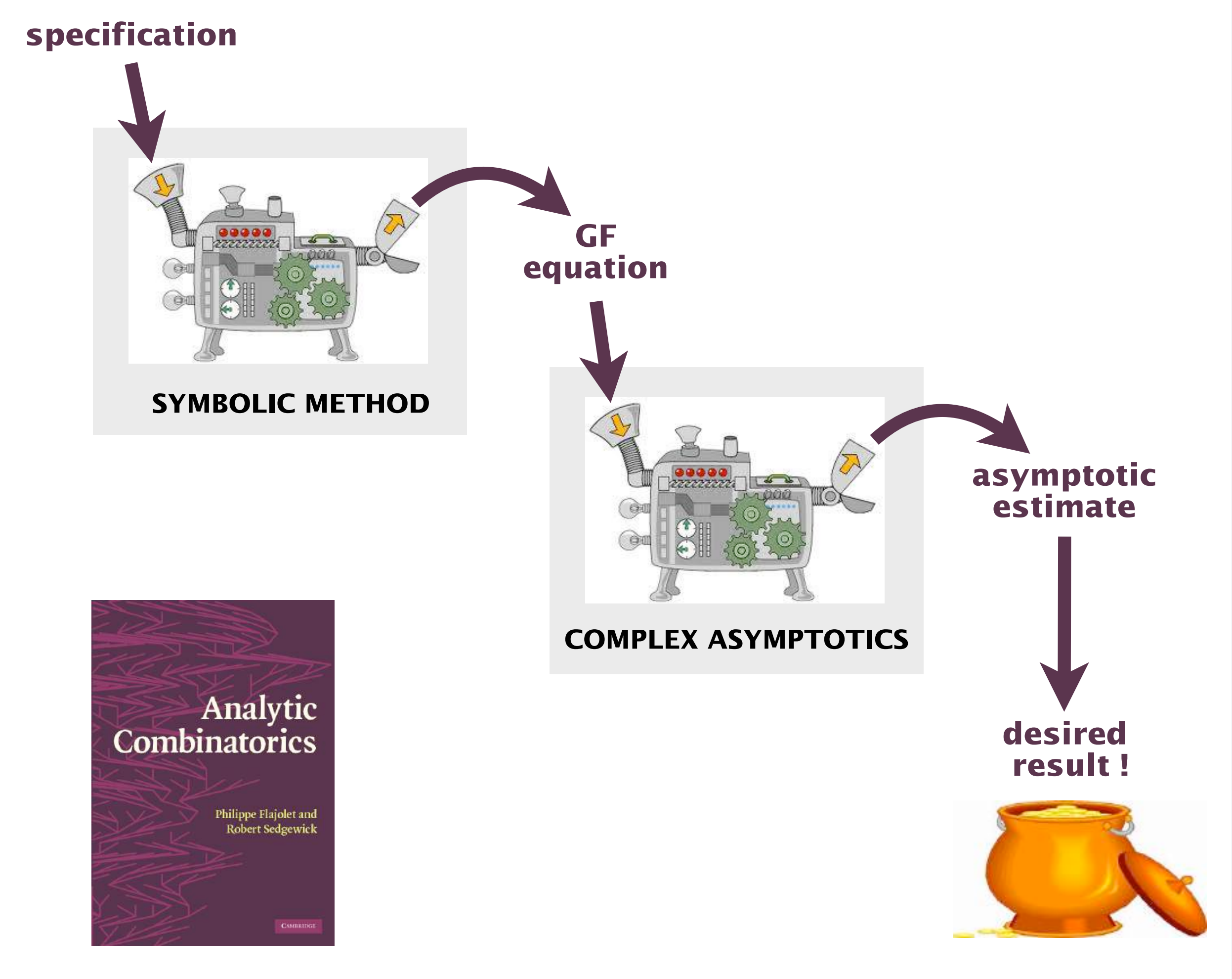
Analytic combinatorics overview

A. SYMBOLIC METHOD

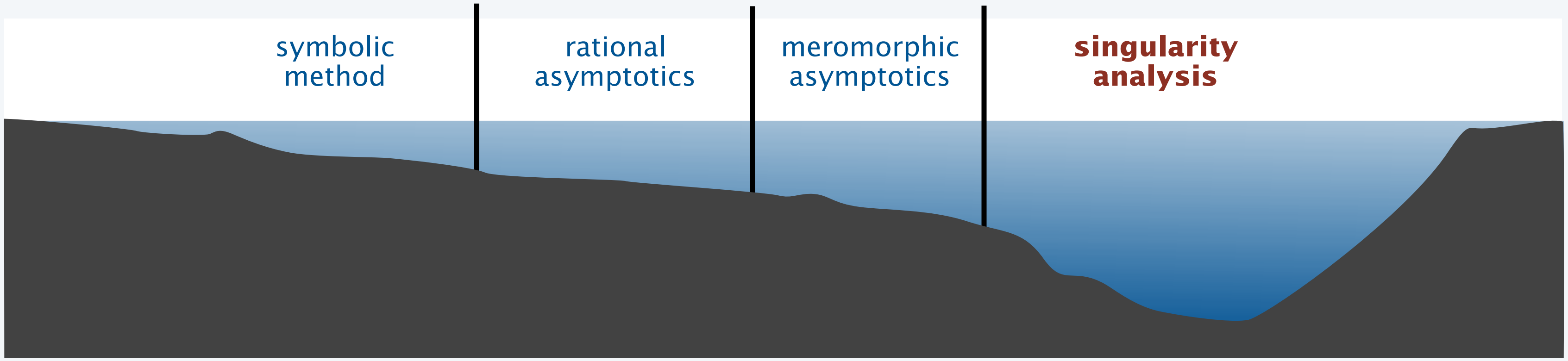
1. OGFs
2. EGFs
3. MGFs

B. COMPLEX ASYMPTOTICS

4. Rational & Meromorphic
5. Applications of R&M
6. Singularity Analysis
7. Applications of SA
8. Saddle point



Warning: entering deep water



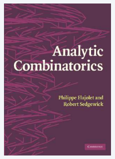
Good news: End results are often broadly applicable and not complicated.

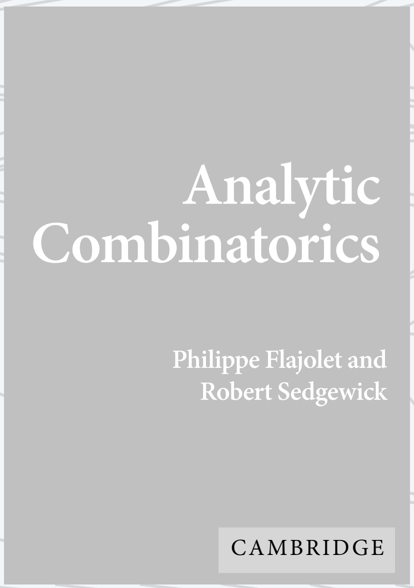
Bad news: Technical skill is often required to *prove* them to be valid

This lecture:

- Overview of approach.
- Statements of several transfer theorems.

For full details refer (always) to The Book.





http://ac.cs.princeton.edu

6. Singularity Analysis

- Prelude
- Standard function scale
- Singularity analysis
- Schemas and transfer theorems

General form of coefficients of combinatorial GFs (revisited)

$$[z^N]F(z) = A^N \theta(N)$$

exponential growth factor

subexponential factor

First principle of coefficient asymptotics

The *location* of a function's singularities dictates the *exponential growth* of its coefficients.

Second principle of coefficient asymptotics

The *nature* of a function's singularities dictates the *subexponential factor* of the growth.

Previous two lectures: $F(z)$ is a meromorphic function $f(z)/g(z)$

- If the smallest real root of $g(z)$ is α then the exponential growth factor is $1/\alpha$.
- If α is a pole of order M, then the subexponential factor is cN^{M-1}.

This lecture: $F(z)$ has singularities that are *not poles*. ← $(1-\alpha z)^M$ is not analytic for any M

Complex square root

Q. Extend the square root function to the complex plane?

Definition. Given $z = re^{i\theta}$ define $\sqrt{z} \equiv \sqrt{r}e^{i\theta/2}$. ← $(\sqrt{z})^2 = z$ ✓

```
public double abs()
{ return Math.hypot(re, im); }

public double phase()
{ return Math.atan2(im, re); }

public Complex sqrt()
{
    double r = Math.sqrt(this.abs());
    double theta = this.phase()/2;
    double x = r*Math.cos(theta);
    double y = r*Math.sin(theta);
    return new Complex(x, y);
}
```

Multiple values problem: $(\sqrt{z})^2 = z$ for infinitely many *z*.

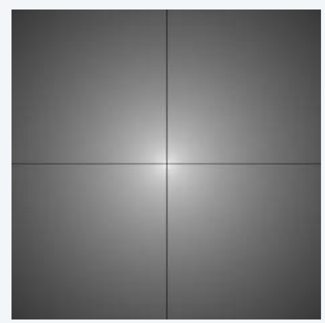
$(\sqrt{r}e^{i\theta/2+k\pi i})^2 = re^{i\theta}$ for any integer k

Definition (revisited). Given $z = re^{i\theta}$ define $\sqrt{z} \equiv \sqrt{r}e^{i\theta/2}$ where $\theta \in (-\pi, \pi]$. ← $\sqrt{z}$ is uniquely defined and $(\sqrt{z})^2 = z$ ✓

Q. Singularities?

A. Yes! But do not show up on absolute value plots *and they are not poles.*

$\sqrt{z}$



Complex square root singularities

Definition. Given $z = re^{i\theta}$ define $\sqrt{z} \equiv \sqrt{r}e^{i\theta/2}$ where $\theta \in (-\pi, \pi]$.

Q. Singularities?

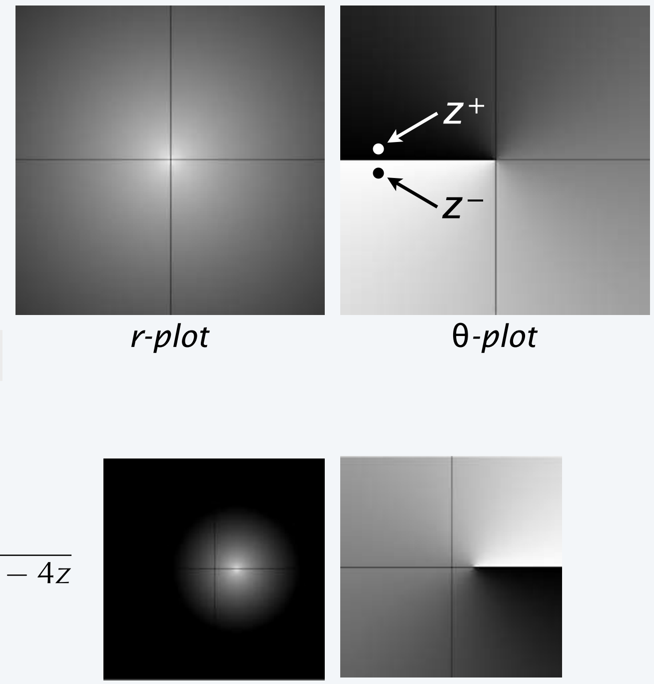
A. Yes, because of discontinuity in the *argument*.

Example:

- Consider the two points $z^+ = -e^{i(\pi-\epsilon)}$ and $z^- = -e^{i(\pi+\epsilon)}$.
- By the definition $\sqrt{z^+} = e^{i(\pi/2-\epsilon/2)}$ and $\sqrt{z^-} = e^{i(-\pi/2+\epsilon/2)}$ $\longleftarrow$ *not* $e^{i(\pi/2+\epsilon/2)}$
- Taking ε arbitrarily small, z^+ and z^- are arbitrarily close together, but the arguments of $\sqrt{z^+}$ and $\sqrt{z^-}$ differ by $i\pi$.
- Therefore, $\sqrt{z}$ is *not differentiable* at $-i$.
- Same argument works for any $z < 0$ on the real line.
- Same argument works by change of variables for any use.

$\sqrt{z}$

$\sqrt{1-4z}$



r-plot θ-*plot*

A. Square root function has an *infinite number* of *essential* singularities.

Complex logarithm

Q. Extend the logarithm function to the complex plane?

Definition. Given $z = re^{i\theta}$ define $\ln z = \ln r + i\theta$. $\leftarrow$ $e^{\ln z} = z$ ✓

```
public Complex log()
{
    double x = Math.log(a.abs());
    double y = a.phase();
    return new Complex(x, y);
}
```

Multiple values problem. $e^{\ln z} = z$ for infinitely many z.

$e^{\ln r + i\theta 2k\pi i} = re^{i\theta}$ for any integer k

Definition (revisited). Given $z = re^{i\theta}$ define $\ln z = \ln r + i\theta$ where $\theta \in (-\pi, \pi]$. $\leftarrow$ $\ln z$ is uniquely defined and $e^{\ln z} = z$ ✓

Other problems.

- $\ln e^z = z$ only when $\theta \in (-\pi, \pi]$
- $\ln wz = \ln w + \ln z$ only when $\theta \in (-\pi, \pi]$

Complex logarithm singularities

Definition. Given $z = re^{i\theta}$ define $\ln z = \ln r + i\theta$ where $\theta \in (-\pi, \pi]$.

Q. Singularities?

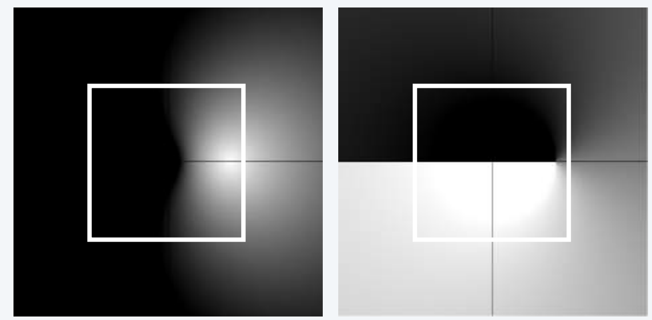
A. Yes, because of discontinuity in the *argument*.

Example:

[omitted, similar to square root example]

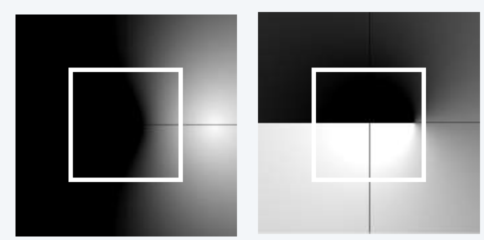
- Same argument works for any $z < 1$ on the real line.
- Same argument works by change of variables for any use.

$\ln z$



r-plot θ-*plot*

$$\frac{1}{1-z}\ln\frac{1}{1-z}$$



A. Logarithm function has an *infinite number* of *essential* singularities.

Gamma function

Q. Extend the factorial function to the complex plane?

Euler representation

$$\Gamma(s) \equiv \int_0^\infty e^{-t} t^{s-1} dt \quad \text{for } \Re(s) > 0$$

Basic identities

$$\Gamma(1) = 1$$

$$\Gamma(s+1) = s\Gamma(s) \qquad s > 0 \text{ (integration by parts)}$$

$$\Gamma(N+1) = N!$$

$$\Gamma(1/2) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt = 2\int_0^\infty e^{-x^2} dx = \sqrt{\pi}$$

$$\Gamma(-\tfrac{3}{2}) = 4\sqrt{\pi}/3$$

$$\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(\tfrac{3}{2}) = \sqrt{\pi}/2$$

$$\Gamma(\tfrac{5}{2}) = \sqrt{\pi}/3$$

Product forms

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^\infty (1 + \frac{s}{n}) e^{-s/n}$$ ← Weierstrass

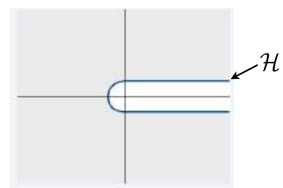
$$\sin s = \prod_{n=1}^\infty (1 - \frac{s^2}{\pi^2 n^2})$$ ← Euler

$$\Gamma(s)\Gamma(-s) = -\frac{\pi}{s \sin \pi s}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Hankel representation

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i} \int_{\mathcal{H}} (-t)^{-s} e^{-t} dt$$



Proof sketch:

$$\frac{1}{2\pi i} \int_{\mathcal{H}} (-t)^{-s} e^{-t} dt = \frac{e^{i\pi s} - e^{-i\pi s}}{2\pi i} \int_0^\infty (-t)^{-s} e^{-t} dt$$

$$= \frac{\sin \pi s}{\pi} \Gamma(1-s) = \frac{1}{\Gamma(s)}$$

Gamma function singularities

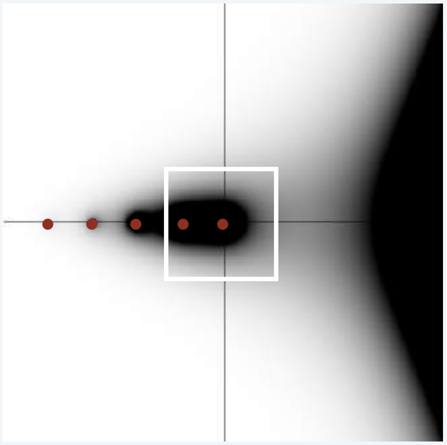
Q. Singularities?

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

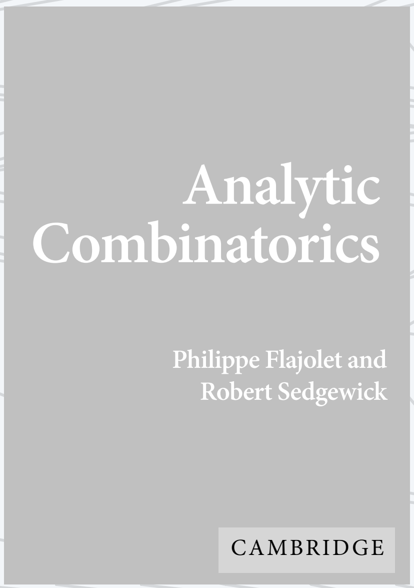
```
public Complex Gamma(Complex z)
{
    double gamma = .5772156649;
    Complex one = new Complex(1.0, 0);
    Complex fact = z.times(z.times(gamma).exp());
    for (int i = 1; i < 10; i++)
    {
       fact = fact.times(one.plus(z.times(1.0/i)));
       fact = fact.times(z.times(-1.0/i).exp());
    }
    return fact.reciprocal();
}
```

A. Yes, simple poles at non-positive integers.

$\Gamma(z)$

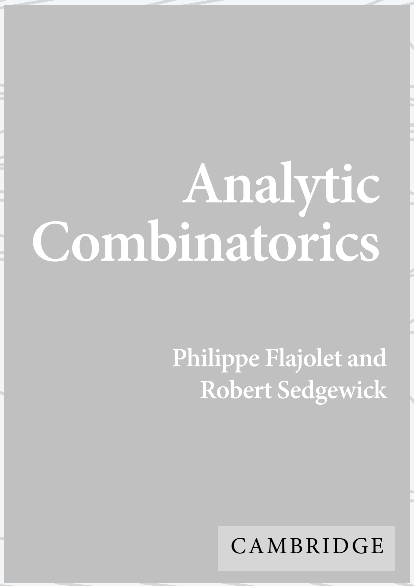


6. Singularity Analysis



http://ac.cs.princeton.edu

- **Prelude**
- Standard function scale
- Singluarity analysis
- Schemas and transfer theorems



http://ac.cs.princeton.edu

6. Singularity Analsis

- Prelude
- **Standard function scale**
- Singularity analysis
- Schemas and transfer theorems

Standard function scale (transfer theorem for non-integral powers)

Theorem. *Standard function scale (leading term).*

For any $\alpha \neq 0, -1, -2, -3, \ldots$ $$[z^N](1-z)^{-\alpha} \sim \frac{N^{\alpha-1}}{\Gamma(\alpha)}$$

Proof .

[See next two slides.]

Extends to give full asymptotic expansion in decreasing powers of N :

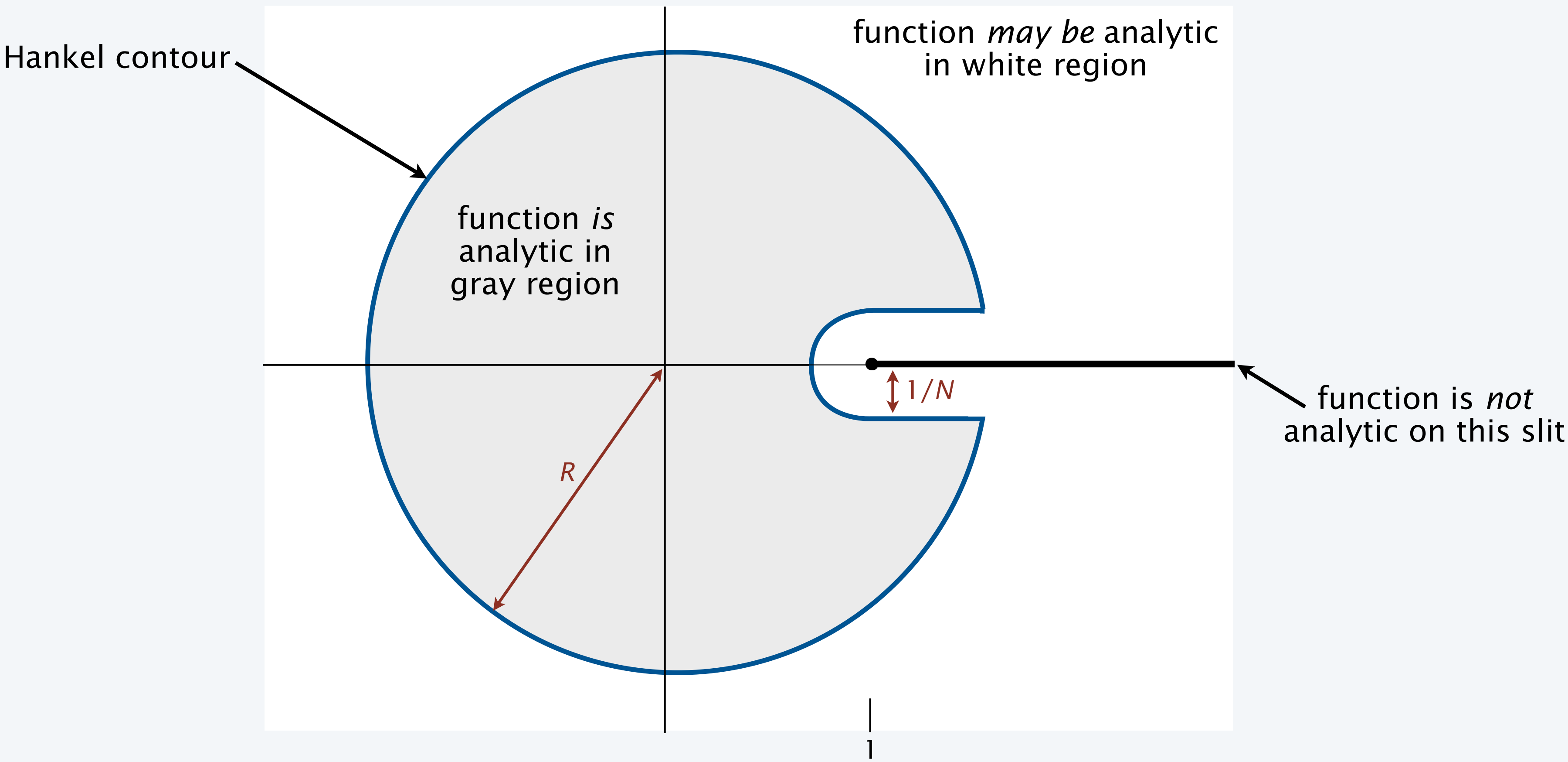
$$[z^N](1-z)^{-\alpha} \sim \frac{N^{\alpha-1}}{\Gamma(\alpha)}\Big(1 + \frac{\alpha(\alpha-1)}{2N} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24N^2} + O\big(\frac{1}{N^3}\big)\Big)$$

Example: $$[z^N]\frac{1-\sqrt{1-4z}}{2} \sim \frac{4^N}{4\sqrt{\pi N^3}}\Big(1 - \frac{3}{8N} - \frac{24}{128N^2} + O\big(\frac{1}{N^3}\big)\Big)$$

$f(z)$	$[z^N]f(z)$
$(1-z)^{3/2}$	$\sim \frac{3}{4\sqrt{\pi N^5}}$
$1-z$	0
$\sqrt{1-z}$	$\sim -\frac{1}{2\sqrt{\pi N^3}}$
1	0
$\frac{1}{\sqrt{1-z}}$	$\sim \frac{1}{\sqrt{\pi N}}$
$\frac{1}{1-z}$	1
$\frac{1}{(1-z)^{3/2}}$	$\sim \frac{2\sqrt{N}}{\sqrt{\pi}}$
$\frac{1}{(1-z)^2}$	$N+1$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Key concept: Hankel contour of radius R and slit width $1/N$



Standard function scale

Theorem. *Standard function scale.* For any $\alpha \neq 0, -1, -2, \ldots$ $\quad [z^N](1-z)^{-\alpha} \sim \dfrac{N^{\alpha-1}}{\Gamma(\alpha)}$

Proof sketch:

- Use Cauchy's coefficient formula for circle C centered at the origin

$$f_N \equiv [z^N](1-z)^{-\alpha} = \frac{1}{2\pi i}\int_C (1-z)^{-\alpha}\frac{dz}{z^{N+1}}$$

- Change of variable $z = 1 + t/N$

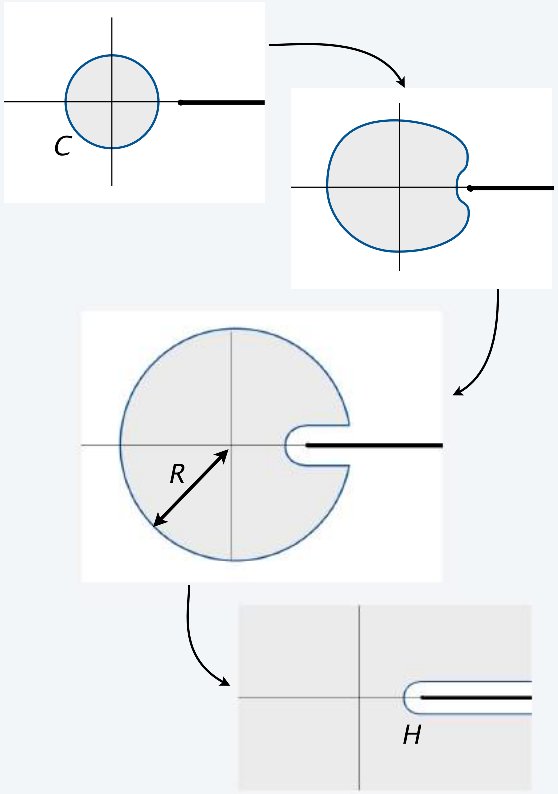
$$[z^N](1-z)^{-\alpha} = \frac{N^{\alpha-1}}{2\pi i}\int_C (-t)^{-\alpha}\Big(1+\frac{t}{N}\Big)^{-N-1}dt$$

- Deform to Hankel contour of radius R and slit width $1/N$

- Take $R \to \infty$

- Apply Hankel's formula for the Gamma function

$$\frac{1}{2\pi i}\int_{\mathcal{H}} (-t)^{-\alpha}\Big(1+\frac{t}{N}\Big)^{-N-1}dt = \frac{1}{\Gamma(\alpha)}$$



Standard function scale with logarithmic factors

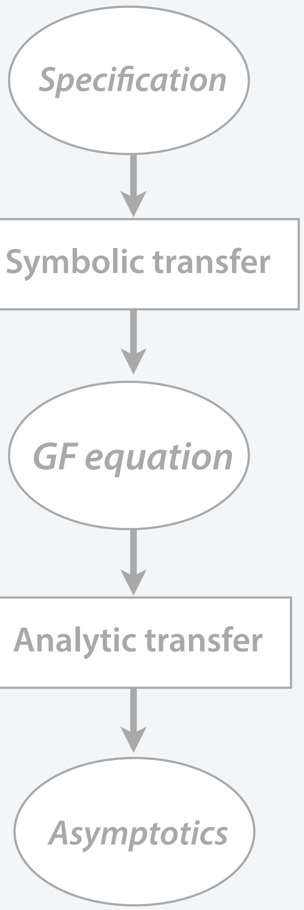
Theorem. For any $\alpha \neq 0, -1, -2, \ldots$ $\quad [z^N]\dfrac{1}{(1-z)^\alpha}\Big(\dfrac{1}{z}\ln\dfrac{1}{1-z}\Big)^\beta \sim \dfrac{N^{\alpha-1}}{\Gamma(\alpha)}(\ln N)^\beta$

Proof sketch:

[omitted, straightforward variant of previous proof]

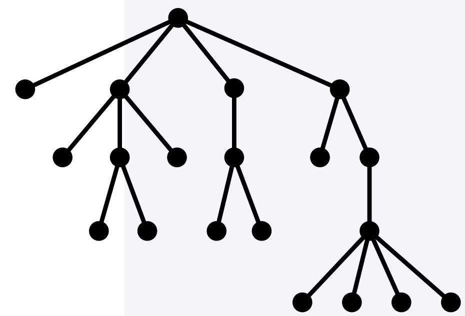
Example: $[z^N]\dfrac{1}{1-z}\ln\dfrac{1}{1-z} \sim \ln N$

AC example with standard scale asymptotics: Binary trees



G, the class of all ordered trees

see Lecture 1

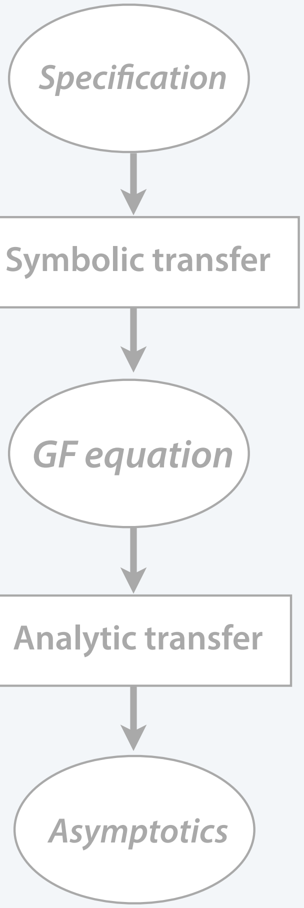


$$\mathbf{G} = \mathbf{Z} \times \mathrm{SEQ}(\mathbf{G})$$

$$\begin{aligned} G(z) &= \frac{z}{1 - G(z)} \\ &= \frac{1 + \sqrt{1-4z}}{2} \end{aligned}$$

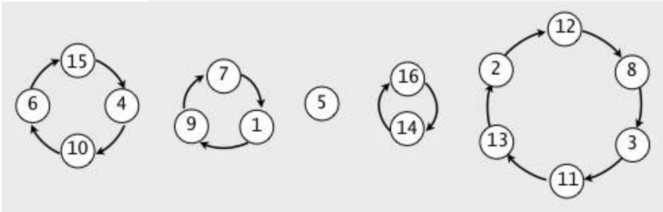
$$[z^N]G(z) \sim \frac{1}{4\sqrt{\pi}} 4^N N^{-3/2}$$

AC example with standard scale asymptotics: Cycles in permutations



P, the class of all permutations ← see Lecture 3

P = SET(**u** CYC(**Z**))



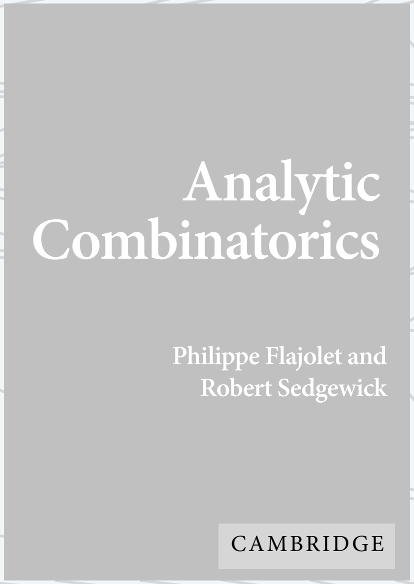
$$P(z,u) = e^{u \ln \frac{1}{1-z}} = (1-z)^{-u}$$

$$P_u(z,1) = \frac{1}{1-z} \ln \frac{1}{1-z}$$ ← OGF for avg number of cycles in a permutation

$$[z^N]P_u(z,1) \sim \ln N$$

ANALYTIC COMBINATORICS

PART TWO



http://ac.cs.princeton.edu

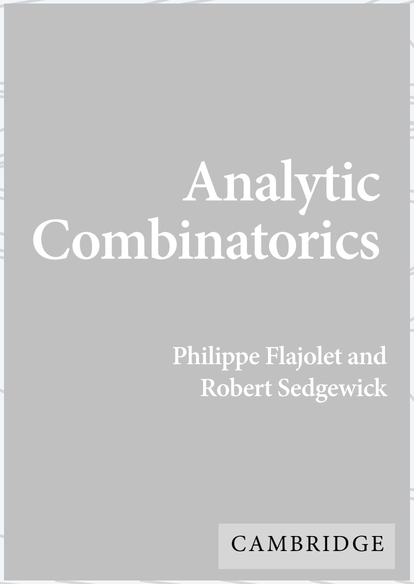
6. Singularity Analysis

- Prelude
- **Standard function scale**
- Singularity analysis
- Schemas and transfer theorems

II.6b.SA.Scale

ANALYTIC COMBINATORICS

PART TWO



http://ac.cs.princeton.edu

6. Singularity Analysis

- Prelude
- Standard function scale
- **Singularity analysis**
- Schemas and transfer theorems

Analytic transfer theorems

Meromorphic ?

- Find dominant pole a, approximate $[z^N]\dfrac{f(z)}{g(z)}$ by $-\dfrac{f(\alpha)}{\alpha g'(\alpha)}\left(\dfrac{1}{\alpha}\right)^N$
- Based on contour integration and residues.

$$D(z) = \frac{e^{-z}}{1-z}$$

Standard function scale ?

- Approximate $[z^N]\dfrac{1}{(1-z)^\alpha}\left(\dfrac{1}{z}\ln\dfrac{1}{1-z}\right)^\beta$ by
- Based on Hankel's representation of the Gamma function.

$$P_u(z,1) = \frac{1}{1-z}\ln\frac{1}{1-z}$$

Neither ?

- Use *singularity analysis.*
- Based on *approximations* to functions in the standard scale.

$$R(z) = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}}$$

No singularities ?

- Use *saddle-point asymptotics.*
- Based on complex analog to Laplace method.

$$I(z) = e^{z+z^2/2}$$

Approximations to functions

Standard approach. Use Taylor theorem to approximate functions at nonsingular points.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f'''(z_0)}{3!}(z - z_0)^3 + \ldots$$

Example:

at $z_0 = 1$

$$e^{-z/2-z^2/4} = e^{-3/4} + e^{-3/4}(1 - z) + \frac{e^{-3/4}}{4}(1 - z)^2 + O(1 - z)^3$$

$$f(z) = e^{-z/2-z^2/4}$$

$$f'(z) = -\frac{1}{2}(1 + z)e^{-z/2-z^2/4}$$

$$f''(z) = (\frac{1}{4}(1 + z)^2 - \frac{1}{2})e^{-z/2-z^2/4}$$

Approximations to functions

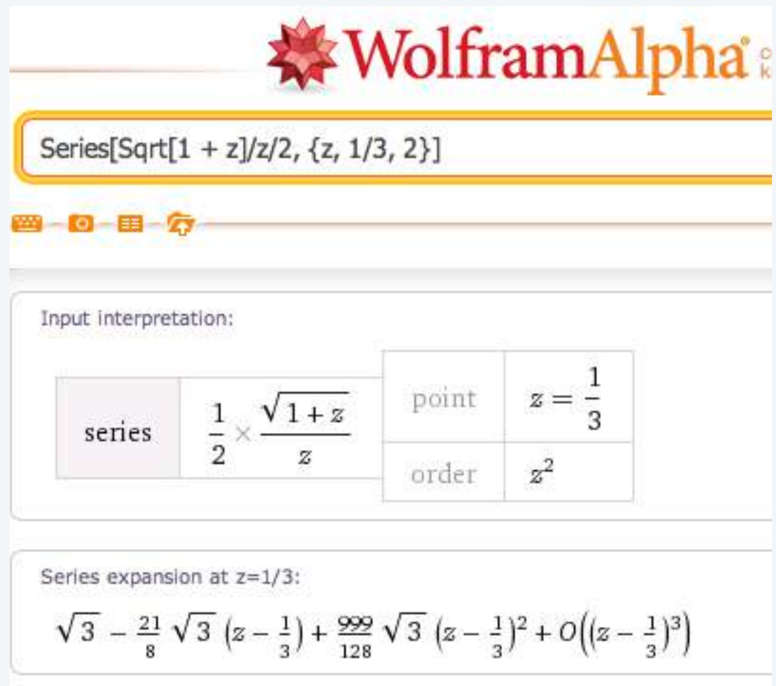
Standard approach. Use Taylor theorem to approximate functions at nonsingular points.

Modern approach. Have a computer do the work!

Example:

at $z_0 = 1/3$

$$\frac{\sqrt{1+z}}{2z} = \sqrt{3} + \frac{7}{8}\sqrt{3}(1-3z) + O((1-3z)^2$$



Singularity analysis (overview)

A general approach to coefficient asymptotics (Flajolet and Odlyzko, 1990).

Locate the singularities.

- Dominant singularity: closest to the origin.
- Location gives the *exponential growth factor*.

Example (unary-binary trees)

$$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}$$

dominant singularity: z= 1/3

exponential growth factor: 3^N

Approximate the function.

- Find domain of analyticity near dominant singularity.
- Use functions from the standard function scale.
- Use approximations that extend (in principle).

first key
to the method

$$M(z) = 1 - \sqrt{3}\sqrt{1-3z} + O(1-3z)^{3/2}$$

Transfer.

- Use known coefficient asymptotics for standard scale.
- *Term-by-term transfer is valid* (!)

second key
to the method

$$M_N = \frac{1}{\sqrt{4\pi/3}} 3^N N^{-3/2} + O(3^N N^{-5/2})$$

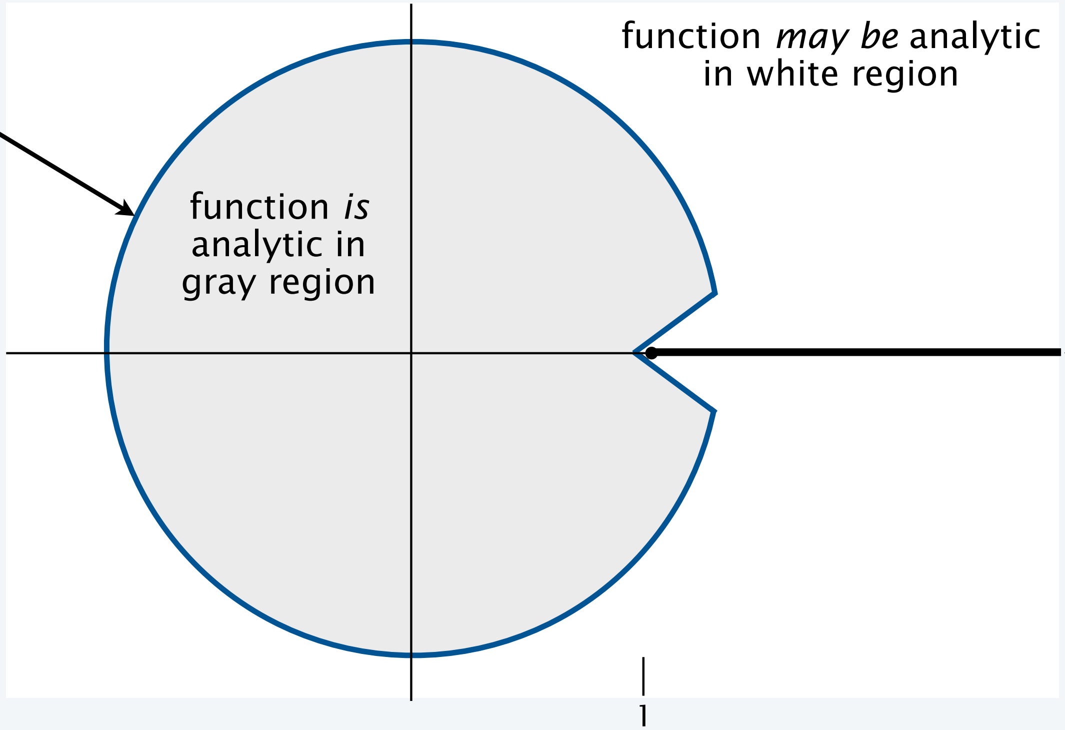
Key concept: Δ-domain

Singularity analysis depends on a function being analytic in a region near its singularities.

Definition. A Δ-*analytic function* is one that is analytic in a Δ-*domain* of the shape depicted below.

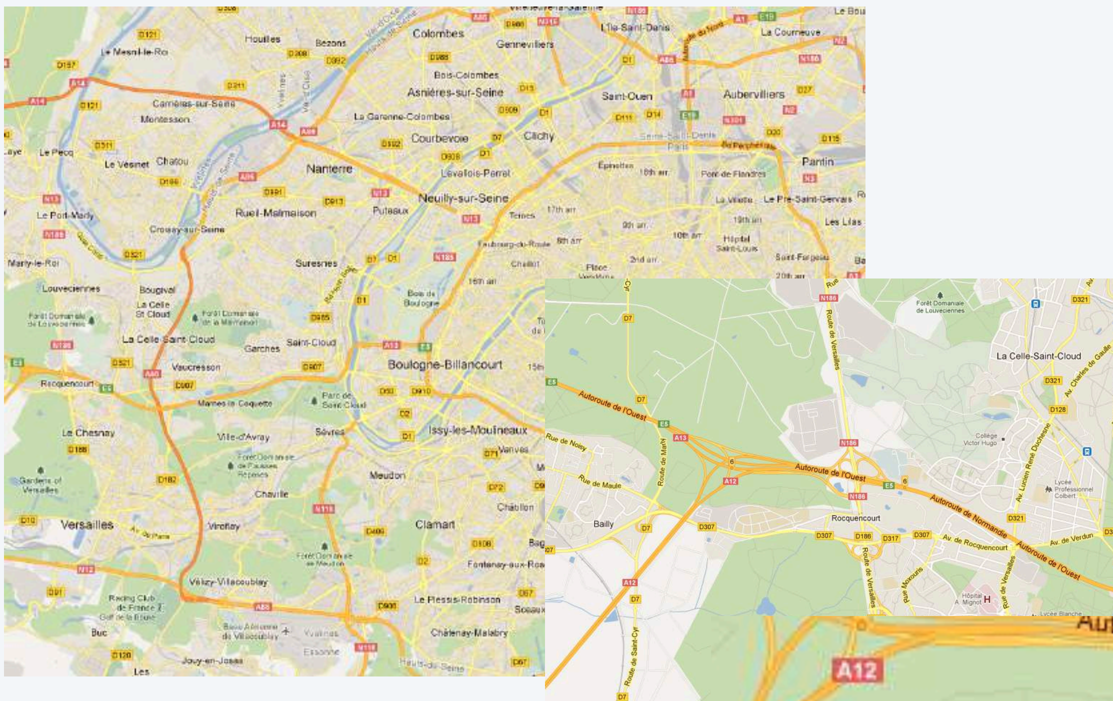
A *weaker* assumption than for the Hankel contour

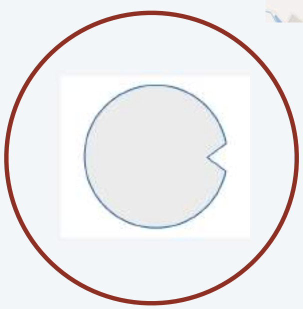
Δ-domain



function is *not* analytic on this slit

Why *that shape* for Δ–domains?





PacMan machine at "corner bar"

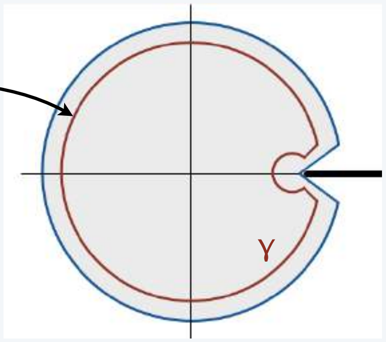
O-transfers, o-transfers, and sim-transfers

Theorem. *O-, o-, and sim-transfers*. Let α and β be real numbers and let $f(z)$ be a Δ-analytic function. Asymptotic approximations of $f(z)$ that hold in the intersection of a neighborhood of 1 with its Δ-domain *transfer* to the corresponding approximations of its coefficients, as follows:

$f(z)$	$O\Big(\dfrac{1}{(1-z)^\alpha}(\ln\dfrac{1}{1-z})^\beta\Big)$	$o\Big(\dfrac{1}{(1-z)^\alpha}(\ln\dfrac{1}{1-z})^\beta\Big)$	$\sim\dfrac{1}{(1-z)^\alpha}(\ln\dfrac{1}{1-z})^\beta$
$[z^N]f(z)$	$O\Big(N^{\alpha-1}(\ln N)^\beta\Big)$	$o\Big(N^{\alpha-1}(\ln N)^\beta\Big)$	$\sim N^{\alpha-1}(\ln N)^\beta$

Brief proof sketch for *O*-transfer.

Use Cauchy's coefficient formula $[z^N]f(z)=\dfrac{1}{2\pi i}\displaystyle\int_\gamma f(z)\frac{dz}{z^{N+1}}$ for this contour

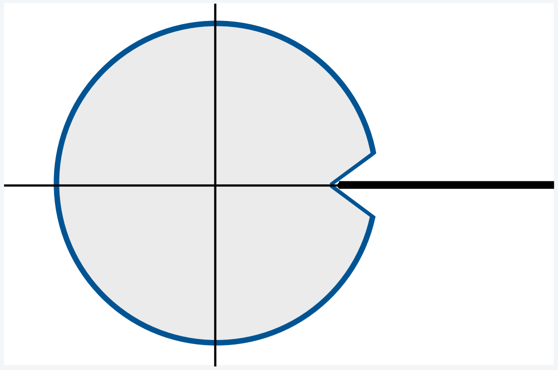


- Small circle: $O\Big(N^{\alpha-1}(\ln N)^\beta\Big)$
- Line segments (the hard part!): $O\Big(N^{\alpha-1}(\ln N)^\beta\Big)$
- Large circle: *exponentially small*

Singularity analysis (summary)

Three steps to coefficient asyptotics for non-meromorphic functions.

1. Preparation.
   - Locate the singularities.
   - Establish analyticity in a Δ-domain around each.



2. Singular expansion.
   - Expand the function near the singularities.
   - Approximate it in the Δ-domain using the standard function scale.

Note: In this lecture, we use sim-transfer.
Key point: Method enables arbitrary asymptotic accuracy.

Transfer.

- Apply O-, o-, and/or sim- transfer theorems.
- Take each term in the function expansion to a term in the asymptotic expansion of its coefficients.

P. Flajolet and A. Odlyzko, *Singularity analysis of generating functions*. SIAM Journal on Algebraic and Discrete Methods **3**, 2 (1990).

Singularity analysis example: Unary-binary trees

Combinatorial class — **M**, the class of all unary-binary trees

Construction — $\mathbf{M} = \bullet \times SEQ_{0,1,2}(\mathbf{M})$

OGF equation — $$M(z) = z(1 + M(z) + M(z)^2)$$

Explicit form — $$M(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}$$

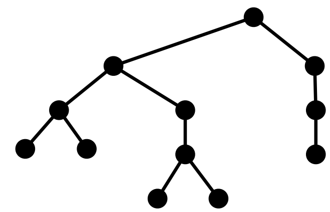
At z = 1/3

$$\frac{\sqrt{1+z}}{2z} = \sqrt{3} + O(1-3z)$$

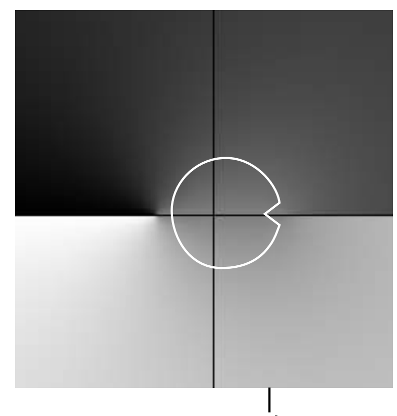
Singular expansion at 1/3 — $$M(z) = 1 - \sqrt{3}\sqrt{1-3z} + O(1-3z)^{3/2}$$

Coefficient asymptotics — $$M_N = \frac{1}{\sqrt{4\pi/3}} 3^N N^{-3/2} + O(3^N N^{-5/2})$$

"a unary-binary tree is a tree where each node has 0, 1, or 2 children"



θ-plot and Δ-domain



1/3

Robustness of singularity analysis

The set of functions amenable to SA is *closed* for natural operations.

under certain technical conditions (as usual)

- Addition.
- Multiplication.
- Composition.
- Differentiation.
- Integration.

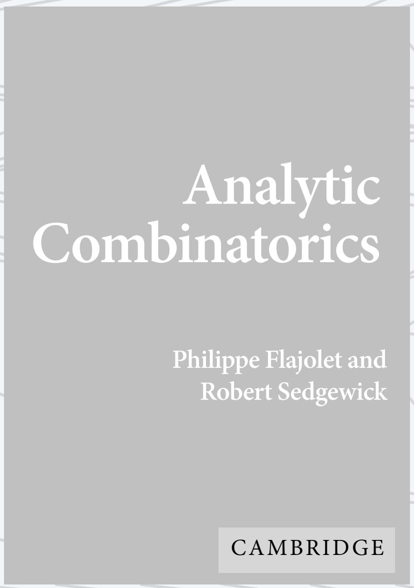
Example: If $f(z)$ and $g(z)$ are Δ-analytic functions then so is $f(z)\ g(z)$.

$$f(z) \sim c(1-z)^{-\alpha} \qquad [z^N]f(z) \sim c\frac{N^{\alpha-1}}{\Gamma(\alpha)}$$

$$g(z) \sim d(1-z)^{-\beta} \qquad [z^N]g(z) \sim d\frac{N^{\beta-1}}{\Gamma(\beta)}$$

$$f(z)g(z) \sim cd(1-z)^{-\alpha-\beta} \qquad [z^N]f(z)g(z) \sim cd\frac{N^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}$$

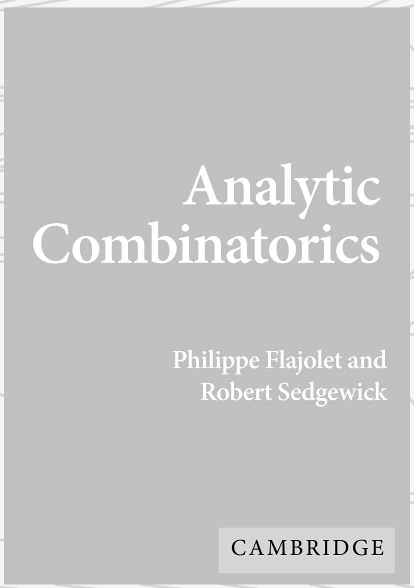
Consequence: GFs produced by the symbolic method are usually amenable to SA



http://ac.cs.princeton.edu

6. Singularity Analysis

- Prelude
- Standard function scale
- **Singularity analysis**
- Schemas and transfer theorems



http://ac.cs.princeton.edu

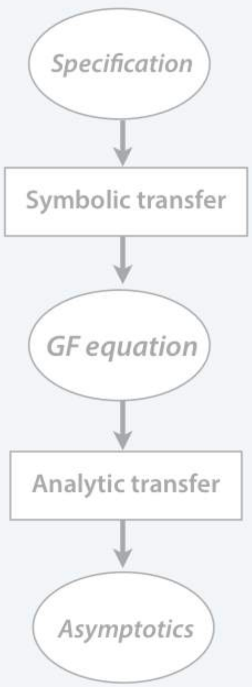
6. Singularity Analysis

- Prelude
- Standard function scale
- Singularity analysis
- **Schemas and transfer theorems**

Schemas

Q. Seems like a lot of work. Any shortcuts?

A. YES. Process is *automatic* for a broad variety of classes.



Recall from previous lecture: A *schema* is a treatment that unifies the analysis of a family of classes.

Next: Examples of schemas that are amenable to singularity analysis (SA):

schema	*technical condition*	*example*	*transfer via*
Sequence	supercritical	$\mathbf{F} = \mathrm{SEQ}(\mathbf{G})$	meromophicity
Labelled set	exp-log	$\mathbf{F} = \mathrm{SET}(\mathbf{G})$	*SA*
Simple variety of trees	invertible	$\mathbf{M} = \bullet \times \mathrm{SEQ}_{0,1,2}(\mathbf{M})$	*SA*
Context-free	irreducible	$\mathbf{S} = \mathbf{E} + \mathbf{U}\times\mathbf{Z}_1\times\mathbf{S} + \mathbf{D}\times\mathbf{Z}_0\times\mathbf{S}$ $\mathbf{U} = \mathbf{Z}_0 + \mathbf{U}\times\mathbf{U}\times\mathbf{Z}_1$ $\mathbf{D} = \mathbf{Z}_1 + \mathbf{D}\times\mathbf{D}\times\mathbf{Z}_0$	*SA*

Schema example 1: Sets

Definition. A labelled class that admits a construction of the form **F** = SET(**G**), where **G** is a labelled class, is said to be a *labelled set class*, which falls within the *labelled set schema*.

Enumeration: **F** = SET(**G**) $\longrightarrow$ $F(z) = e^{G(z)}$

$$f_N = [z^N]F(z)$$
$$g_N = [z^N]G(z)$$

labelled: number of structures is $N!\,f_N$

Parameters:

mark number of **G** components with u

F = SET(u **G**) $\longrightarrow$ $F(z,u) = e^{uG(z)}$

mark number of **G**$_k$ components with u

F = SET(u **G**$_k$ + **G**\ **G**$_k$) $\longrightarrow$ $F^k(z,u) = e^{(u-1)g_k z^k} F(z)$

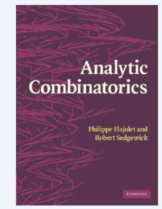
Labelled exp-log classes

exp-log: A technical condition that enables us to unify the analysis of labelled set classes.

Definition. *Exp-log labelled set classes.*

A labelled set class **F** = SET(**G**) is said to be *exp-log*(α, β, ρ) if the EGF *G*(*z*) associated with **G** satisfies the following conditions:

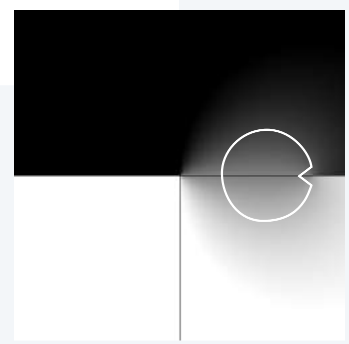
- G(z) is analytic at 0 and has nonnegative coefficients.
- G(z) has finite radius of convergence ρ.
- The number ρ is the unique singularity of G(z) on |z| = ρ.
- G(z) is continuable to a Δ-domain at ρ.
- As z → ρ in Δ $G(z) \sim \alpha \log \dfrac{1}{1 - z/\rho} + \beta$



page 446

Example: GF for cycles: $Y(z) = \ln \dfrac{1}{1-z}$

analytic except for real z > 1 and z < 0



Therefore, the class of permutations **P** = SET(**Y**) is exp-log(1, 0, 1).

Transfer theorem for exp-log labelled set classes

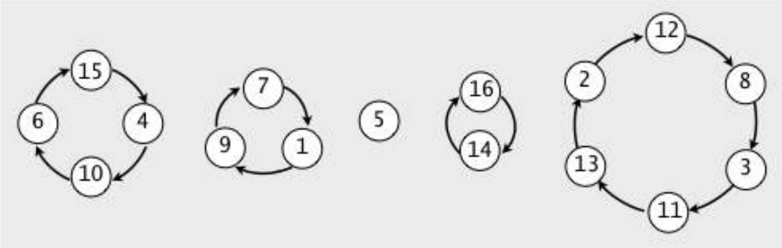
Theorem. *Asymptotics of exp-log labelled sets.*

Suppose that a labelled set class **F** = SET$_\Phi$(**G**) is exp-log(α, β, ρ) with $G(z) \sim \alpha \log \dfrac{1}{1 - z/\rho} + \beta$. Then $F(z) \sim e^{\beta}\Big(\dfrac{1}{1 - z/\rho}\Big)^{\alpha}$

and $[z^N]F(z) \sim \dfrac{e^{\beta}}{\Gamma(\alpha)}\Big(\dfrac{1}{\rho}\Big)^{N} N^{1-\alpha}$

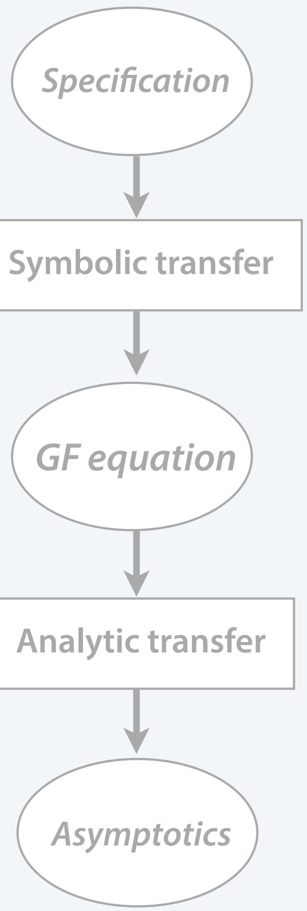
Corollary. The expected number of *G*-components in a random *F*-object of size *N* is ~α ln *N*.

and is concentrated there



Brief proof sketch: Check all the conditions; apply SA

AC example with exp-log labelled set schema asymptotics: Cycles in permutations



P, the class of all permutations

P = SET(CYC(**Z**))

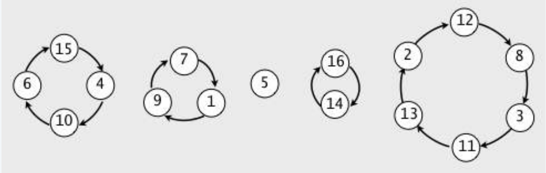
$$P(z) = \exp(\ln \frac{1}{1-z})$$

$$[z^N]P(z) \sim 1$$

avg # permutations: $\sim N$

avg # cycles: $\sim \ln N$

Next lecture: Many more examples



Theorem. *Asymptotics of exp-log labelled sets.*

Suppose that a labelled set class **F** = SET$_\Phi$(**G**) is exp-log(α, β, ρ) with $G(z) \sim \alpha \log \frac{1}{1-z/\rho} + \beta$. Then $F(z) \sim e^{\beta}\left(\frac{1}{1-z/\rho}\right)^{\alpha}$

and $[z^N]F(z) \sim \frac{e^{\beta}}{\Gamma(\alpha)}\left(\frac{1}{\rho}\right)^N N^{1-\alpha}$

$$\ln \frac{1}{1-z} = \alpha \log \frac{1}{1-z/\rho} + \beta$$

for $\alpha = 1, \beta = 0,$ and $\rho = 1$

Corollary. The expected number of *G*-components in a random *F*-object of size *N* is ~α ln *N*.

and is concentrated there

Schema example 2: Simple varieties of trees

Definition. A combinatorial class whose enumeration GF satisfies $F(z) = z\phi(F(z))$ is said to be a *simple variety of trees* with *characteristic function* ϕ.

Examples:

unlabelled case: number of structures is $[z^N]F(z)$

$\mathbf{F} = \mathbf{Z} \times \mathrm{SEQ}_\Omega(\mathbf{F})$

$\mathbf{F} = \mathbf{Z} \times \mathrm{SET}_\Omega(\mathbf{F})$

labelled case: number of structures is $N![z^N]F(z)$

$\mathbf{F} = \mathbf{Z} \star \mathrm{SEQ}_\Omega(\mathbf{F})$

$\mathbf{F} = \mathbf{Z} \star \mathrm{SET}_\Omega(\mathbf{F})$

$$F(z) = z\phi(F(z))$$

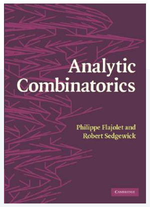
all *immediate*
via symbolic transfer

Invertible tree classes

invertible: A technical condition that enables us to unify the analysis of tree classes.

Definition. *Invertible tree classes.* A simple variety of trees whose GF satisfies $F(z) = z\phi(F(z))$ is said to be λ-*invertible* if its characteristic function $\phi(u)$ satisfies the following conditions:

- $\phi(u)$ has nonnegative coefficients, and is *not* of the form $\phi_0 + \phi_1 u$.
- $\phi(u)$ is analytic at 0 with $\phi(0) \neq 0$ and radius of convergence R.
- The characteristic equation $\phi(\lambda) = \lambda\phi'(\lambda)$ has a positive real real root $\lambda < R$.



page 453

Example: Rooted ordered trees

Construction	$\mathbf{G} = \mathbf{Z} \times \mathrm{SEQ}(\mathbf{G})$
OGF equation	$G(z) = \dfrac{z}{1 - G(z)}$
Characteristic function	$\phi(u) = \dfrac{1}{1-u}$
Characteristic equation	$\dfrac{1}{1-u} = \dfrac{u}{(1-u)^2}$
positive real root	$\lambda = 1/2$

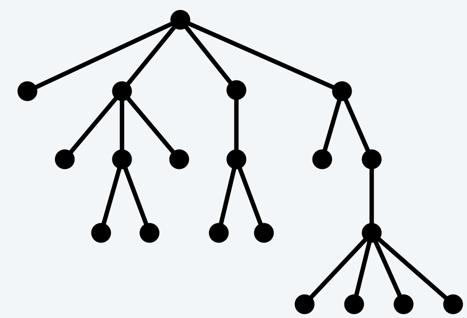
$$\phi'(u) = \frac{1}{(1-u)^2}$$

Trees are 1/2-*invertible*

Transfer theorem for simple varieties of trees

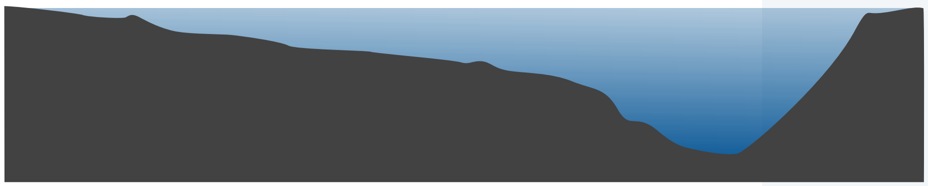
Theorem. If a simple variety of trees with GF $F(z) = z\phi(F(z))$ is λ-invertible (where λ is the positive real root of $\phi(u) = u\phi'(u)$) then

$$[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}}\Big(\phi'(\lambda)\Big)^N N^{-3/2}$$



Proof approach.

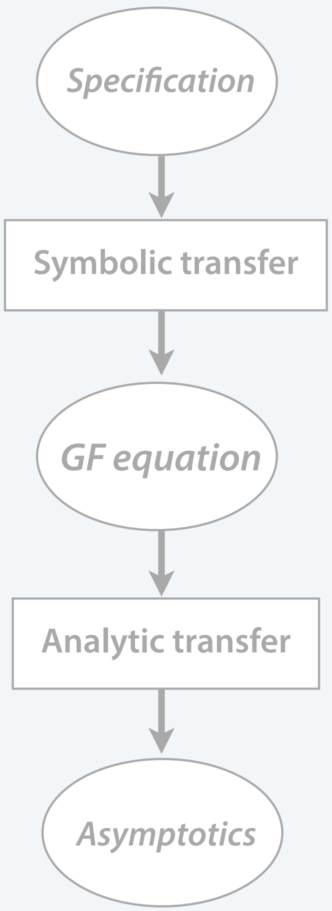
1. Use *analytic inversion* to show that
$F(z) \sim \lambda - \sqrt{2\phi(\lambda)/\phi''(\lambda)}\sqrt{1 - z\phi'(\lambda)}$
2. Transfer via standard function scale.



Surprising fact: $N^{-3/2}$ factor is present for *all* simple varieties of trees.

Note: "periodic" ϕ introduce complications that we ignore in lecture (see text).

AC example with invertible tree schema asymptotics: Rooted ordered trees

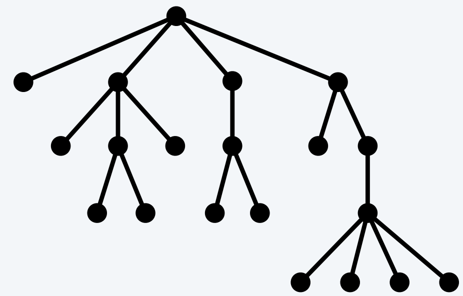


G, the class of rooted ordered trees

$$G = Z \times \mathrm{SEQ}(G)$$

$$G(z) = \frac{z}{1 - G(z)}$$

$$G_N \sim \frac{1}{4\sqrt{\pi}} 4^N N^{3/2}$$



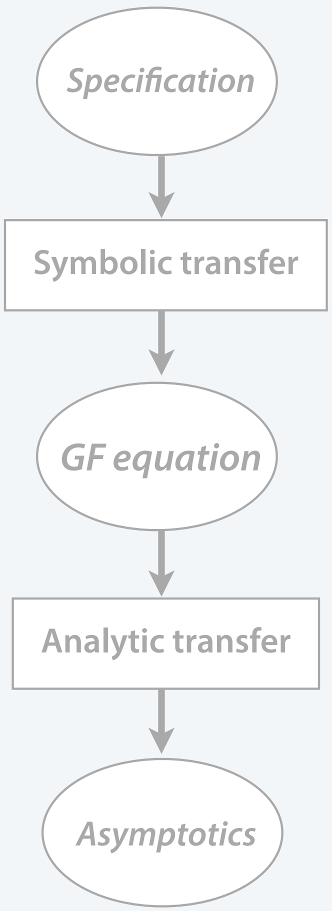
Theorem. If a simple variety of trees with GF $F(z) = z\phi(F(z))$ is λ-invertible (where λ is the positive real root of $\phi(u) = u\phi'(u)$) then $[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}}\Big(\phi'(\lambda)\Big)^N N^{-3/2}$

$$\phi(u) = \frac{1}{1-u}$$
$$\phi'(u) = \frac{1}{(1-u)^2}$$
$$\phi''(u) = \frac{1}{(1-u)^3}$$

$$\frac{1}{1-\lambda} = \frac{\lambda}{(1-\lambda)^2}$$

$$\lambda = 1/2$$
$$\phi(\lambda) = 2$$
$$\phi'(\lambda) = 4$$
$$\phi''(\lambda) = 16$$

AC example with invertible tree schema asymptotics: Unary-binary trees

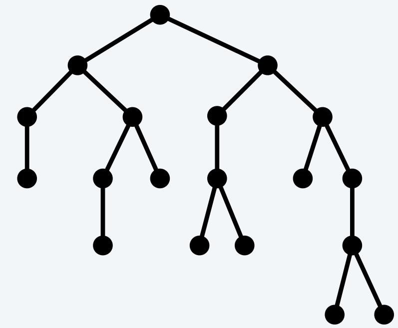


M, the class of all unary-binary trees

$$\mathbf{M} = \mathbf{Z} \times SEQ_{0,1,2}(\mathbf{M})$$

$$M(z) = z(1 + M(z) + M(z)^2)$$

$$M_N \sim \frac{1}{\sqrt{4\pi/3}} 3^N N^{-3/2}$$



Theorem. If a simple variety of trees with GF $F(z) = z\phi(F(z))$ is λ-invertible (where λ is the positive real root of $\phi(u) = u\phi'(u)$) then $[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}}\bigl(\phi'(\lambda)\bigr)^N N^{-3/2}$

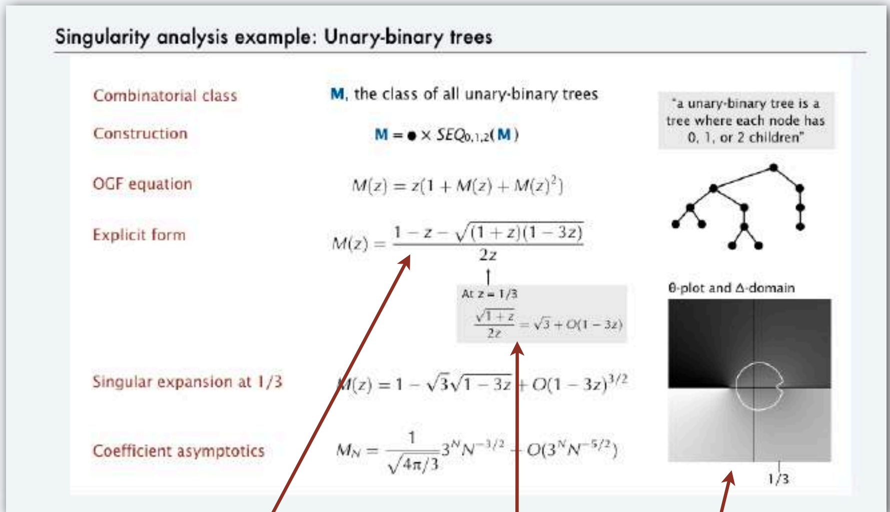
$$\phi(u) = 1 + u + u^2$$
$$\phi'(u) = 1 + 2u$$
$$\phi''(u) = 2$$

$$1 + \lambda + \lambda^2 = \lambda + 2\lambda$$

$$\lambda = 1$$
$$\phi(\lambda) = 3$$
$$\phi'(\lambda) = 3$$
$$\phi''(\lambda) = 2$$

Next lecture: Many more examples

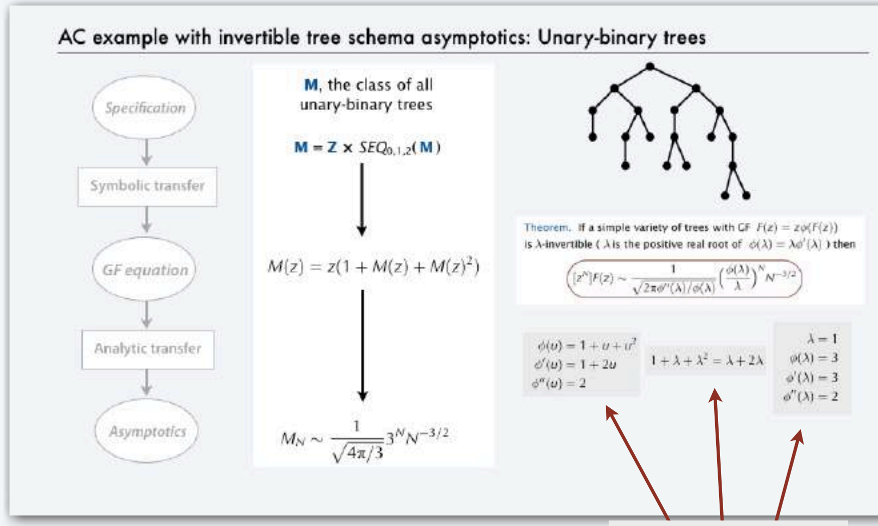
Significance of tree schema



Need to solve polynomial equation

Need to check analyticity

Need to expand



"plug and chug"

The schema *unifies the analysis* for an entire family of classes.

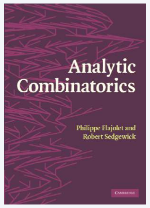
- Compute the exponential growth (from the characteristic function).
- Compute the constant (from the characteristic function).
- Surprising fact: $N^{-3/2}$ factor is present for all simple varieties of trees.

Schema example 3: Context-free classes

Definition. A combinatorial class that admits a construction of the form

$$\mathbf{Y} = \mathbf{Y_1} = CONSTRUCT\,(\mathbf{Z}, \mathbf{Y_1}, \mathbf{Y_2}, \ldots, \mathbf{Y_r})$$
$$\mathbf{Y_2} = CONSTRUCT\,(\mathbf{Z}, \mathbf{Y_1}, \mathbf{Y_2}, \ldots, \mathbf{Y_r})$$
$$\vdots$$
$$\mathbf{Y_r} = CONSTRUCT\,(\mathbf{Z}, \mathbf{Y_1}, \mathbf{Y_2}, \ldots, \mathbf{Y_r})$$

where *CONSTRUCT* is a construction that involves only $+$ and $\times$, is said to be a *context-free class*, which falls within the *context-free schema*.



page 483

Example: Strings with equal numbers of 0s and 1s.

$$\mathbf{S} = \mathbf{E} + \mathbf{U}\times\mathbf{Z}_1\times\mathbf{S} + \mathbf{D}\times\mathbf{Z}_0\times\mathbf{S}$$
$$\mathbf{U} = \mathbf{Z}_0 + \mathbf{U}\times\mathbf{U}\times\mathbf{Z}_1$$
$$\mathbf{D} = \mathbf{Z}_1 + \mathbf{D}\times\mathbf{D}\times\mathbf{Z}_0$$

$$\overset{\mathbf{U}}{\overline{0010100010111}}\,1\,\underset{\mathbf{D}}{\underline{10111000}}\,0\,\underset{\mathbf{D}}{\underline{11010}}\,0$$

Interpretation:
- **U** is the set of strings where # of 0s > # of 1s in any prefix.
- **D** is the set of strings where # of 1s > # of 0s in any prefix.

Irreducible context-free classes

irreducible: A technical condition that enables us to unify the analysis of context-free classes.

Definition. *Irreducible context-free classes.* A context-free class is said to be *irreducible* if it is nonlinear and its dependency graph is strongly connected.

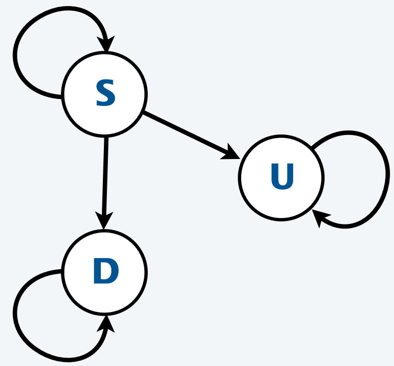
Example: Strings with equal numbers of 0s and 1s.

$$\mathbf{S} = \mathbf{E} + \mathbf{U}\times\mathbf{Z}_1\times\mathbf{S} + \mathbf{D}\times\mathbf{Z}_0\times\mathbf{S}$$

$$\mathbf{U} = \mathbf{Z}_0 + \mathbf{U}\times\mathbf{U}\times\mathbf{Z}_1$$

$$\mathbf{D} = \mathbf{Z}_1 + \mathbf{D}\times\mathbf{D}\times\mathbf{Z}_0$$

nonlinear



not strongly connected

Irreducible context-free classes

irreducible: A technical condition that enables us to unify the analysis of context-free classes.

Definition. *Irreducible context-free classes*. A context-free class is said to be *irreducible* if it is nonlinear and its dependency graph is strongly connected.

Example: "Non-crossing forests".

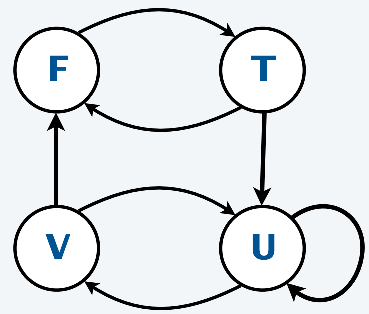
$$\mathbf{F} = \mathbf{E} + \mathbf{T}$$

$$\mathbf{T} = \mathbf{Z} \times \mathbf{F} \times \mathbf{U}$$

$$\mathbf{U} = \mathbf{E} + \mathbf{U} \times \mathbf{V}$$

$$\mathbf{V} = \mathbf{Z} \times \mathbf{F} \times \mathbf{U} \times \mathbf{U}$$

nonlinear



strongly connected

Transfer theorem for irreducible context-free classes

Theorem. If C is an irreducible context-free class, then its generating function $C(z)$ has a square-root singularity at its radius of convergence ρ. If $C(z)$ is aperiodic, then the dominant singularity is unique and $[z^N]F(z) \sim \frac{1}{\sqrt{\alpha\pi}}\left(\frac{1}{\rho}\right)^N N^{-3/2}$ where α is a computable real.

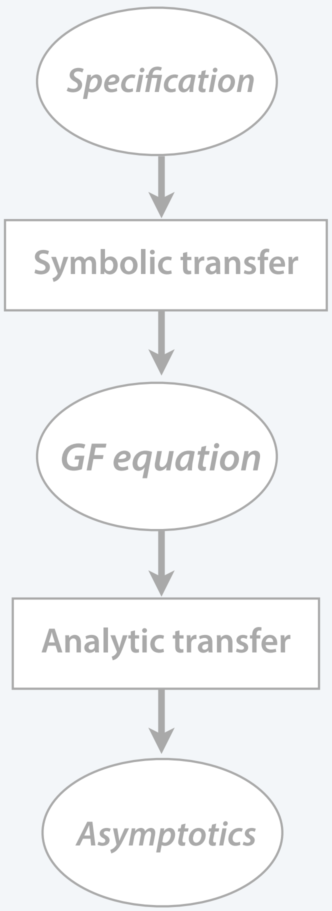
Proof approach.

Drmota-Lalley-Woods theorem.

Computing the constant?

- Can be complicated.
- Maybe best left for a computer.

"If you can specify it, you can analyze it"



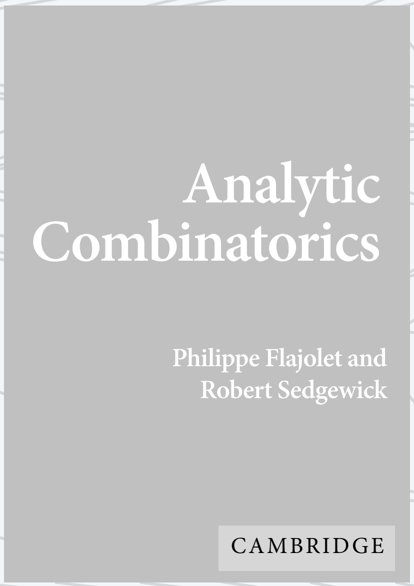
Singularity analysis is an effective approach to develop analytic transfer from GF equations to coefficient asymptotics for combinatorial classes.

Analysis can be detailed and burdensome.

Schema can unify the analysis for entire families of classes.

schema	*technical condition*	*construction*	*coefficient asymptotics*
Labelled set	exp-log	$\mathbf{F} = \mathrm{SET}(\mathbf{G})$	$\dfrac{e^\beta}{\Gamma(\alpha)}\left(\dfrac{1}{\rho}\right)^N N^{1-\alpha}$
Simple variety of trees	invertible	$\mathbf{F} = \mathbf{Z} \times \mathit{SEQ}(\mathbf{F})$ $\mathbf{F} = \mathbf{Z} \star \mathit{SEQ}(\mathbf{F})$	$\dfrac{1}{\sqrt{\alpha\pi}}\left(\dfrac{1}{\rho}\right)^N N^{-3/2}$
Context-free	irreducible	Family of (+, X) constructs	$\dfrac{1}{\sqrt{\alpha\pi}}\left(\dfrac{1}{\rho}\right)^N N^{-3/2}$

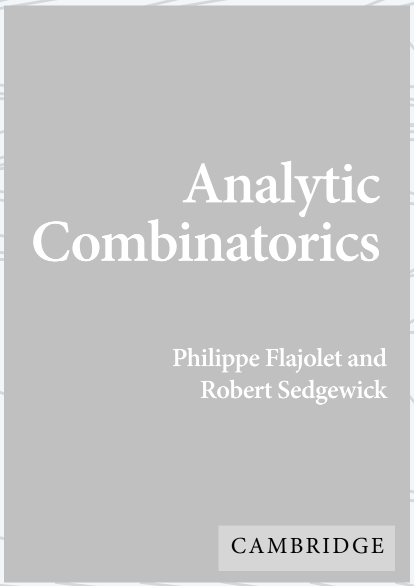
Note: Several other schemas have been developed (stay tuned).



http://ac.cs.princeton.edu

6. Singularity Analysis

- Prelude
- Standard function scale
- Singularity analysis
- **Schemas and transfer theorems**



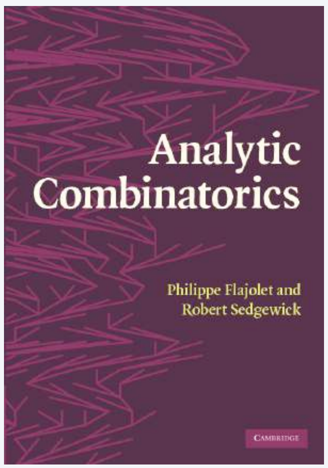
http://ac.cs.princeton.edu

6. Singularity Analysis

- Prelude
- Standard function scale
- Singularity analysis
- Schemas and transfer theorems
- **Exercises**

Web Exercise VI.1

Standard scale.



Web Exercise VI.1. Use the standard function scale to directly derive an asymptotic expression for the number of strings in the following CFG:

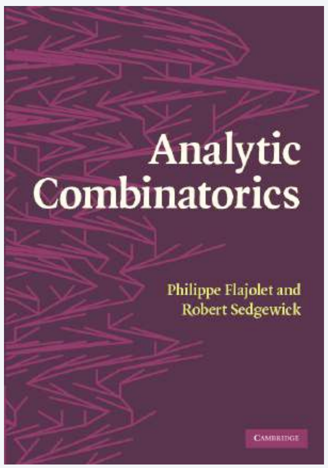
$$\mathbf{S} = \mathbf{E} + \mathbf{U}\times\mathbf{Z}_1\times\mathbf{S} + \mathbf{D}\times\mathbf{Z}_0\times\mathbf{S}$$

$$\mathbf{U} = \mathbf{Z}_0 + \mathbf{U}\times\mathbf{U}\times\mathbf{Z}_1$$

$$\mathbf{D} = \mathbf{Z}_1 + \mathbf{D}\times\mathbf{D}\times\mathbf{Z}_0$$

Web Exercise VI.2

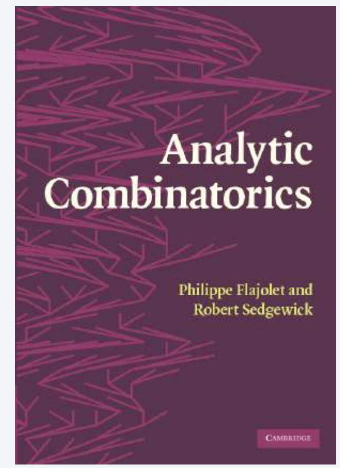
2-3 trees (of a certain type)



Web Exercise VI.2. Give an asymptotic expression for the number of rooted ordered trees for which every node has 0, 2, or 3 children. How many bits are necessary to represent such a tree?

Assignments

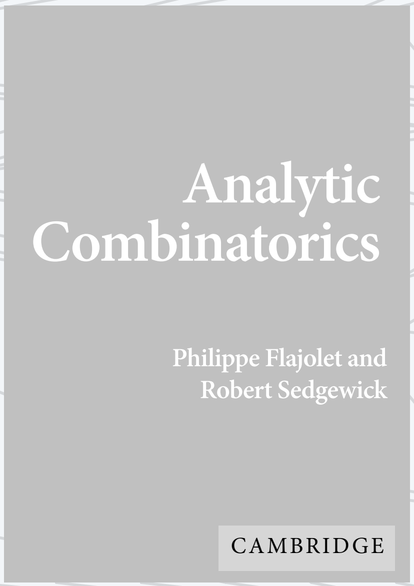
1. Read pages 375-438 (*Singularity Analysis of Generating Functions*) in text.
   Usual caveat: Try to get a feeling for what's there, not understand every detail.



2. Write up solutions to Web exercises VI.1 and VI.2.

3. Programming exercise.

Program VI.1. Do *r*- and θ-plots of $1/\Gamma(z)$ in the unit square of size 10 centered at the origin.



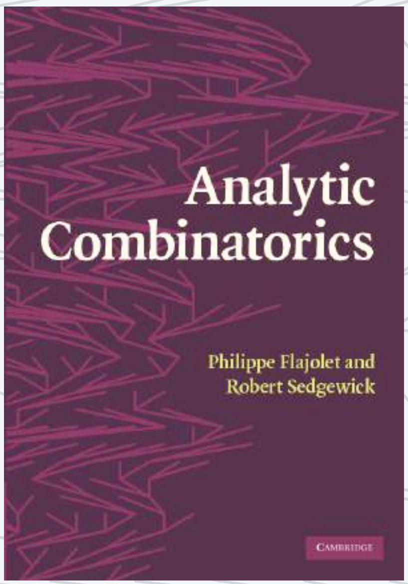
http://ac.cs.princeton.edu

6. Singularity Analysis

- Prelude
- Standard function scale
- Singularity analysis
- Schemas and transfer theorems
- **Exercises**

ANALYTIC COMBINATORICS

PART TWO



http://ac.cs.princeton.edu

6. Singularity Analysis