ANALYTIC COMBINATORICS

PART TWO

Analytic Combinatorics

Philippe Flajolet and Robert Sedgewick

8. Saddle-Point Asymptotics

http://ac.cs.princeton.edu

Analytic combinatorics overview





Warmup: 2D absolute value plots

Consider 2D plots of functions: all points (x, |f(x)|) in a Cartesian plot.



Welcome to absolute-value-land!

Consider 3D versions of our plots of analytic functions. A *modulus surface* is a plot of (x, y, |f(z)|) where z = x + yi.



Q. Can a modulus surface assume any shape ?

A. No.

A. (A surprise.) Only four types of points.

Modulus surface points type I: zeros

A *zero* is a point where f(z) = 0 and $f'(z) \neq 0$.



Ex. $f(z) = 2z = 2re^{i\theta}$, |f(z)| = 2r

A zero of order p is a point where $f^{(k)}(z) = 0$ for $0 \le k < p$ and $f^{(p)}(z) \ne 0$.



Modulus surface points type II: poles



Quick in-class exercise

Q. What function is this?





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Modulus surface points type III: ordinary points

An ordinary point is a point where $f(z) \neq 0$ and $f'(z) \neq 0$.

All ordinary points have the same local behavior.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \qquad \sim c$$



Modulus surface points type III: saddle points

A saddle point is a point where $f(z) \neq 0$ and f'(z) = 0.

All saddle points have the same local behavior.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \quad (z - z_0)^2$$



Basic characteristic

- Downwards-oriented parabola at one angle
- Upwards-oriented parabola at perpendicular angle



Modulus surface points: summary



Quick in-class exercise

Q. Where are the saddle points?



Modulus surface plots for familiar AC GFs







Saddle-point bound for GFs: basic idea



Saddle-point bounds for GFs

Theorem. Saddle point bounds for GFs.

Let G(z), not a polynomial, be analytic at the origin with finite radius of convergence R. If G has nonnegative coefficients, then $[z^N]G(z) \le G(\zeta)/\zeta^N$ where ζ is the *saddle point* closest to the origin, the unique real root of the *saddle point equation* $\zeta G'(\zeta)/G(\zeta) = N + 1$.

С

Proof (sketch). By Cauchy coefficient formula

$$[z^{N}]G(z) = \frac{1}{2\pi i} \int_{C} G(z) \frac{dz}{z^{N+1}}$$

$$= \frac{\zeta}{2\pi} \int_{0}^{2\pi} G(z) \frac{d\theta}{z^{N+1}}$$
Take *C* to be a circle of radius ζ and change to polar coordinates
$$\leq \frac{G(\zeta)}{\zeta^{N}}$$
 $G(z) \leq G(\zeta)/\zeta^{N+1}$ on

Example:
$$[z^5]e^z$$

 $G(z) = e^z$
 $G'(z) = e^z$
 $\zeta = 6$
 $[z^N]e^z = \frac{1}{5!} \le \frac{e^6}{6^5}$
 $\doteq .008333$
 $\doteq .009498$

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Saddle point GF bound example I: factorial/exponential

Goal. Estimate
$$\frac{1}{N!} = [z^N]e^z$$
 $G(z) = e^z$
Saddle point equation
 $z \frac{e^z}{e^z} = N + 1$
Saddle pointSaddle point equation
 $z = N + 1$ Saddle point equation
 $z \frac{G'(z)}{G(z)} = N + 1$ Saddle point $\zeta = N + 1$ Saddle point boundSaddle point bound
 $[z^N]e^z = \frac{1}{N!} \le \frac{e^{N+1}}{(N+1)^N}$
 $\rightarrow \frac{e^N}{N^N}$ Saddle point bound
 $[1 + \frac{1}{N})^N \rightarrow e$ Bound is too high by only a factor of $\sqrt{2\pi N}$, since $\frac{1}{N!} \sim \frac{e^N}{N^N\sqrt{2\pi N}}$

Saddle point GF bound example II: Catalan/central binomial

Goal. Estimate
$$\binom{2N}{N} = [z^N](1+z)^{2N}$$
 $G(z) = (1+z)^{2N}$ Saddle point equation $2Nz = (N+1)(1+z)$ Saddle point $2Nz = (N+1)(1+z)$ $z = \frac{G'(z)}{G(z)} = N+1$ Saddle point $\zeta = \frac{N+1}{N-1}$ $z^{2N(1+z)^{2N-1}} = N+1$ Saddle point bound $\binom{2N}{N} \leq \frac{\binom{2N}{N-1}^{2N}}{\binom{N+1}{N-1}^N} = \binom{4N^2}{N^2-1}^N$ Saddle point boundSaddle point bound $\binom{2N}{N} \leq \frac{\binom{2N}{N-1}^{2N}}{\binom{N+1}{N-1}^N} = \binom{4N^2}{N^2-1}^N$ Saddle point boundBound is too high by only a factor of $\sqrt{\piN}$, since $\binom{2N}{N} \sim \frac{4^N}{\sqrt{\piN}}$

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Saddle-point *method* for GFs: basic idea



Saddle-point susceptibility

susceptibility: Technical conditions that enable us to unify saddle-point approximations.

Definition. Saddle-point susceptible contour integrals.

The contour integral $\int_C F(z)dz$ with $F(z) = e^{f(z)}$ is susceptible to the saddle point approximation if C passes through a saddle point ζ , the unique real root of the saddle point equation F'(z) = 0 (or f'(z) = 0) and C can be split into two parts T and Q such that

• Tails are negligable:
$$\int_T F(z)dz = o\left(\int_C F(z)dz\right)$$

• A central quadratic approximation holds uniformly along *Q*: $f(z) \sim f(\zeta) + \frac{1}{2}f''(\zeta)(z-\zeta)^2$

• Tails can be completed back [details omitted].

 ζ to be expected unless multiple saddle point since f'(ζ)= 0

Saddle-point transfer theorem

Theorem. If a contour integral $\int_C F(z)dz$ with $F(z) = e^{f(z)}$ is susceptible to the saddle point approximation, then $\frac{1}{2\pi i} \int_C F(z)dz \sim \frac{F(\zeta)}{\sqrt{2\pi f''(\zeta)}}$

a general technique for contour integration (not just for asymptotics)

Proof.

[Similar to proof for SP bound; see text]

Saddle-point transfer theorem

Theorem. If a contour integral $\int_C F(z)dz$ with $F(z) = e^{f(z)}$ is susceptible to the saddle point approximation, then $\frac{1}{2\pi i}\int_C F(z)dz \sim \frac{F(z)}{\sqrt{2\pi f''(\zeta)}}$

Saddle-point transfer. Given a GF G(z), if the contour integral of $G(z)/z^{N+1}$ along a path C is susceptible to the saddle point approximation, then

$$[z^{N}]G(z) = \frac{1}{2\pi i} \int_{C} G(z) \frac{dz}{z^{N+1}} \sim \frac{e^{g(\zeta)}}{\sqrt{2\pi g''(\zeta)}}$$

where $g(z) = \ln G(z) - (N + 1) \ln z$ and ζ is the unique positive real root of the saddle point equation g'(z) = 0.

Equivalent forms

 $\frac{G'(z)}{G(z)} = \frac{N+1}{z}$

SP approximation $G(\zeta) \over \overline{\zeta^{N+1} \sqrt{2\pi g''(\zeta)}}$

Proof. Take $F(z) = G(z)/z^{N+1}$.

Saddle point transfer example I: factorial/exponential

Coal. Estimate
$$\frac{1}{N!} = [z^N]e^z$$
 $G(z) = e^z$ $f(z) = \ln G(z) - (N+1) \ln z$ Saddle point $\zeta = N+1$ $f'(z) = 1 - \frac{N+1}{z}$ $f'(z) = 0$ $f''(z) = \frac{N+1}{z^2}$ $f'(z) = 0$ Saddle point $[z^N]e^z = \frac{1}{N!} \sim \frac{e^{N+1}}{(N+1)^{N+1}\sqrt{2\pi/(N+1)}}$ $[z^N]G(z) \sim \frac{G(\zeta)}{\zeta^{N+1}\sqrt{2\pi f''(\zeta)}}$ $\sim \frac{e^N}{N^N\sqrt{2\pi N}}$ \checkmark $(1 + \frac{1}{N})^N \rightarrow e$

Important note: Need to check susceptibility, or use bound and sacrifice $\sqrt{2\pi N}$ factor.

tails are negligible, a central approximation holds, and tails can be completed back

Saddle point method example I (susceptibility to saddle point)

Contour integral

$$\frac{1}{N!} = [z^{N}]e^{z} = \frac{1}{2\pi i} \int_{C_{N}} e^{z} \frac{dz}{z^{N+1}} = \frac{1}{2\pi i} \int_{C_{N}} e^{z-(N+1)\ln z} dz$$
Switch to polar coordinates

$$= \left(\frac{e}{N}\right)^{N} \frac{1}{2\pi} \int_{0}^{2\pi} e^{N(e^{i\theta}-1-i\theta)} d\theta$$
Split into central and tail contours

$$= \frac{1}{2\pi} \left(\frac{e}{N}\right)^{N} (Q_{N} + T_{N})$$

$$Q_{N} = \int_{-\theta_{0}}^{+\theta_{0}} e^{N(e^{i\theta}-1-i\theta)} d\theta$$

$$T_{N} = \int_{\theta_{0}}^{2\pi-\theta_{0}} e^{N(e^{i\theta}-1-i\theta)} d\theta$$
exponentially small for

$$\theta_{0} = N^{\kappa} \text{ with } \alpha > -1/2 [\text{ see text}]$$

Note: Slightly shifting saddle point (from N+1 to N) simplifies calculations.

Saddle point method example I (susceptibility to saddle point)

$$G_{N} = \int_{-\theta_{0}}^{+\theta_{0}} e^{N(e^{i\theta} - 1 - i\theta)} d\theta$$
Approximate integrand
$$= \int_{-\theta_{0}}^{+\theta_{0}} e^{-N\theta^{2}/2} d\theta \left(1 + O(N\theta_{0}^{3})\right) \qquad (e^{i\theta} - 1 - i\theta) = -\theta^{2}/2 + O(\theta^{3})$$
Restrict θ_{0} to drop *O*-term
$$\sim \int_{-\theta_{0}}^{+\theta_{0}} e^{-N\theta^{2}/2} d\theta \quad \text{for } \theta_{0} = N^{\alpha} \text{ with } \alpha < -1/3$$
Change of variable
$$\sim \frac{1}{\sqrt{N}} \int_{-\theta_{0}\sqrt{N}}^{+\theta_{0}\sqrt{N}} e^{-t^{2}/2} dt \qquad \theta = t/\sqrt{N}$$
d $\theta = dt/\sqrt{N}$
Restrict θ_{0} to complete tails
$$\sim \frac{1}{\sqrt{N}} \int_{-\infty}^{+\infty} e^{-t^{2}/2} dt \quad \text{for } \theta_{0} = N^{\alpha} \text{ with } \alpha > -1/2$$
Collect restrictions
$$\sim \sqrt{2\pi/N} \quad \text{for } \theta_{0} = N^{2/5} \qquad \int_{N^{1/2-\alpha}}^{\infty} e^{-t^{2}/2} dt = O(e^{-N^{1-2\alpha}})$$
Finish
$$\frac{1}{N!} \sim \frac{1}{2\pi} \left(\frac{\theta}{N}\right)^{N} G_{N} = \left(\frac{\theta}{N}\right)^{N} \frac{1}{\sqrt{2\pi N}} \checkmark$$

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Saddle-point asymptotics



- A. Methods extend to derive full asymptotic series to any desired precision.
- A. Results are easy to validate numerically.
- A. Towards goal of general schema cover whole families of combinatorial classes.

Saddle point transfer example II: Catalan/central binomial

Goal. Estimate
$$\binom{2N}{N} = [z^N](1+z)^{2N}$$
 $f(z) = \ln G(z) - (N+1) \ln z$
 $= 2N \ln(1+z) - (N+1) \ln z$
 $= 2N \ln(1+z) - (N+1) \ln z$
 $f'(z) = \frac{2N}{1+z} - \frac{N+1}{z}$ Saddle point equation
 $f'(z) = 0$ Saddle point
approximation $[z^N](1+z)^{2N} = \binom{2N}{N} \sim \frac{\left(\frac{2N}{N-1}\right)^{2N}}{\left(\frac{N+1}{N-1}\right)^{N+1}\sqrt{2\pi f''(\frac{N+1}{N-1})}}$ Saddle point approx
 $\frac{[z^N]G(z) \sim \frac{G(\zeta)}{\zeta^{N+1}\sqrt{2\pi f''(\zeta)}}$

Note: Slight shift of saddle point often simplifies calcuations (see next slide).

Saddle point transfer example II: Catalan/central binomial

Goal. Estimate
$$\binom{2N}{N} = [z^N](1+z)^{2N}$$
 $f(z) = \ln G(z) - (N+1) \ln z$
 $= 2N \ln(1+z) - (N+1) \ln z$
 $= 2N \ln(1+z) - (N+1) \ln z$
 $f'(z) = \frac{2N}{1+z} - \frac{N+1}{z}$ Saddle point equation
 $f'(z) = 0$ Saddle point
approximation $[z^N](1+z)^{2N} = \binom{2N}{N} \sim \frac{4^N}{\sqrt{\pi N}}$ $Saddle point approx $(z^N)G(z) \sim \frac{G(\zeta)}{\zeta^{N+1}\sqrt{2\pi f''(\zeta)}}$$

Important note: Need to check susceptibility, or use bound and sacrifice $\sqrt{\pi N}$ factor.

tails are negligible, a central approximation holds, and tails can be completed back





Involutions

Q. How many different permutations of size *N* with no cycle lengths >2 ?



AC example with saddle-point asymptotics: Involutions





Saddle-point transfer. Given a GF *G*(*z*), if the contour integral $\frac{1}{2\pi i} \int_C G(z) \frac{dz}{z^{N+1}}$ is susceptible to the saddle point approximation, then

$$\left([z^N]G(z) = \frac{1}{2\pi i} \int_C G(z) \frac{dz}{z^{N+1}} \sim \frac{\mathrm{e}^{g(\zeta)}}{\sqrt{2\pi g''(\zeta)}} \right)$$

where $g(z) = \ln G(z) - (N + 1) \ln z$ and ζ is the unique positive real root of the saddle point equation g'(z) = 0 (equivalently, G'(z)/G(z) = (N+1)/z).

$$g(z) = z + z^{2}/2 - (N+1) \ln z$$

$$g'(z) = 1 + z - \frac{N+1}{z}$$

$$\zeta^{2} + \zeta - (N+1) = 0$$

$$\zeta^{2} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4(N+1)}$$

$$\sim \sqrt{N} - \frac{1}{2} + O(\frac{1}{\sqrt{N}})$$

Important note: Need to check susceptibility.

- generally more difficult than for other transfer thms.
- option: use bound (sacrifice $\sqrt{2\pi N}$ factor.

Set partitions

	{1} {2} {3} {4}				
Q. How many ways to	{1} {2 3 4}				
				$\{2\} \{1 \ 3 \ 4\}$ $\{3\} \{1 \ 2 \ 4\}$ $\{4\} \{1 \ 2 \ 3\}$	
$\{1\}$ $S_1 = 1$	${1}$ {2} {1 2} $S_2 = 2$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$		$\begin{array}{cccccccccccccccccccccccccccccccccccc$	

AC example with saddle-point asymptotics: Set partitions



$$e^{e^{z}-1}/z^{6}$$

$$\{2 \ 3\} \ \{5 \ 7 \ 9\} \ \{4\} \ \{1 \ 8\}$$

$$(2 \ 3) \ \{5 \ 7 \ 9\} \ \{4\} \ \{1 \ 8\}$$

$$(3 \ 2) \ (3 \$$

$$\left([z^N]G(z) = \frac{1}{2\pi i} \int_C G(z) \frac{dz}{z^{N+1}} \sim \frac{e^{g(\zeta)}}{\sqrt{2\pi g''(\zeta)}} \right)$$

where $g(z) = \ln G(z) - (N + 1) \ln z$ and ζ is the unique positive real root of the saddle point equation g'(z) = 0 (equivalently, G'(z)/G(z) = (N+1)/z).

$$g(z) = e^{z} - 1 - (N+1) \ln z$$

$$g'(z) = e^{z} - \frac{N+1}{z}$$

$$\zeta e^{\zeta} = N+1$$

$$\zeta \sim \ln N - \ln \ln N$$

SP bound

$$[z^N]S(z) \le \frac{G(\zeta)}{\zeta^N}$$

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Saddle point: summary of combinatorial applications

	construction	GF	saddle point bound	coefficient asymptotics	
urns	$\mathbf{U} = SET(\mathbf{Z})$	e ^z	$ ightarrow rac{e^N}{N^N}$	$\frac{1}{N!} \sim \frac{e^N}{N^N \sqrt{2\pi N}}$	
central binomial		$[z^N](1+z)^{2N}$	$\rightarrow 4^N$	$\sim rac{4^N}{\sqrt{\pi N}}$	
involutions	$\mathbf{I} = SET(CYC_{1,2}(\mathbf{Z}))$	$e^{z+z^{2}/2}$	$\leq N! \frac{e^{N/2 + \sqrt{N} - 1/4}}{\sqrt{2}N^{N/2}}$	$\sim N! rac{\mathrm{e}^{N/2 + \sqrt{N} - 1/4}}{2N^{N/2}\sqrt{\pi N}}$	
set partitions	$\mathbf{S} = SET(SET_{>0}(\mathbf{Z}))$	e^{e^z-1}	$\leq N! \frac{e^{N-1}}{(\ln N)^N}$	*	
fragmented permutations	$\mathbf{F} = SET(SEQ_{>0}(\mathbf{Z}))$	$e^{z/(1-z)}$	$\leq N! e^{2\sqrt{N}-1/2}$	$\leq N! \frac{e^{2\sqrt{N}-1/2}}{2\sqrt{\pi}N^{3/4}}$	not for amateur
integer partitions	$\mathbf{P} = MSET(SEQ_{>0}(\mathbf{Z}))$	$e^{z/(1-z)+z^2/2(1-z^2)+}$	$\leq e^{\pi \sqrt{2N/3}}$	$\sim rac{e^{\pi\sqrt{2N/3}}}{4N\sqrt{3}}$	





Analytic combinatorics overview



1. Combinatorial specifications provide succinct definitions of a wide range of discrete structures.

- 2. The *symbolic method* transforms specifications to equations that define *generating functions*.
- 3. *Complexification* treats generating functions as *analytic* objects, giving estimates of coefficients.

Cauchy's coefficient formula gives coefficient asymptotics when singularities are poles.

Singularity analysis provides a general approach to analyzing GFs with essential singularities.

Saddle-point asymptotics is effective for functions with no singularities.

4. Combinatorial classes fall into general *schema* that are governed by *universal* asymptotic laws.

Constructions and symbolic transfers

operation	notation	semantics		OGF			
disjoint union	A + B	disjoint copies of objects from A and B	A(z) + B(z)				
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from <i>A</i> and one from <i>B</i>	A(z)B(z)				
sequence	SEQ(A)	sequences of objects from A					
				The symbolic metho	d for labelled cla	usses (transfer theorem)	
powerset	PSET(A)	finite sets of objects from A (no repetitions)	$\prod_{n\geq 1}(1+z)$	Theorem. Let A and	<i>B</i> be combinatoria	l classes of labelled objects with EGF	s A(z) and E
multiset	MSET(A)	finite sets of objects from A	Ππ	construction	notation	semantics	EGF
		(with repetitions)	<u>n≥1</u> (1	disjoint union	A + B	disjoint copies of objects from A and B	A(z) + B(z)
dditional co	nstructs ar	e available (and still being invented)	—one exa	labelled product	A ★ B	ordered pairs of copies of objects, one from A and one from B	A(z)B(z)

sequence

set

cycle

SEQ(A)

 $SET_k(A)$

SET(A)

 $CYC_k(A)$

CYC(A)

4	2
т	5

1

 $\overline{1 - A(z)}$

 $A(z)^k/k!$

 $e^{A(z)}$

 $A(z)^k/k$

 $\ln \frac{1}{1-A(z)}$

sequences of objects from A

k-sets of objects from A

sets of objects from A

k-cycles of objects from A

cycles of objects from A

Explicit analytic transfers



Schemas

Combinatorial problems can be organized into broad schemas, covering infinitely many combinatorial types and governed by simple asymptotic laws.

Theorem. Asymptotics of exp-log labelled sets. Suppose that a labelled set class $\mathbf{F} = SET_{\Phi}(\mathbf{G})$ is exp-log(α , β , ρ) with $G(z) \sim \alpha \log \frac{1}{1-z/\rho} + \beta$. Then $F(z) \sim e^{\beta} \left(\frac{1}{1-z/\rho}\right)^{\alpha}$ $(z^N]F(z) \sim \frac{e^{\beta}}{\Gamma(\alpha)} (\frac{1}{\rho})^N N^{1-\alpha}$ and

Theorem. If C is an irreducible context-free class, then its generating function C(z) has a square-root singularity at its radius of convergence ρ . If C(z) is aperiodic, then the dominant singularity is unique and $(z^N)F(z) \sim \frac{1}{\sqrt{\alpha\pi}}(\frac{1}{\rho})^N N^{-3/2}$ where α is a computable real.

The discovery of such schemas and of the associated universality properties constitues the *very essence* of analytic combinatorics.

Theorem. Asymptotics of supercritical sequences. If $\mathbf{F} = SEQ(\mathbf{G})$ is a strongly aperiodic supercritical sequence class, then $(z^N)F(z) \sim \frac{1}{C'(\lambda)} \frac{1}{\lambda^{N+1}}$ where λ is the root of $G(\lambda) = 1$ in $(0, \rho)$. radius of convergence of G(z)

Theorem. If a simple variety of trees with GF $F(z) = z\phi(F(z))$ is λ -invertible (where λ is the positive real root of $\phi(u) = u\phi'(u)$) $[z^N]F(z) \sim \frac{1}{\sqrt{2\pi\phi''(\lambda)/\phi(\lambda)}} \Big(\phi'(\lambda)\Big)^N N^{-3/2}$ then

Theorem. Asymptotics of implicit tree-like classes. Suppose that **F** is an implicit tree-like class with associated GF $F(z) = \Phi(z, F(z))$ that is aperiodic and smooth-implicit(r, s), so that G(r, s) = s and $G_w(r, s) = 1$. Then F(z) converges at z = r where it has a square-root singularity with $F(z) \sim s - \alpha \sqrt{1 - z/r} \operatorname{and}\left([z^N]F(z) \sim \frac{\alpha}{2\sqrt{\pi}} (\frac{1}{r})^N N^{-3/2}\right)$ where $\alpha = \sqrt{\frac{2r\Phi_z(r,s)}{\Phi_{WW}(r,s)}}$.

"If you can specify it, you can analyze it"



[In case someone asks...]

Analytic combinatorics aims to enable precise quantitative predictions of the properties of large combinatorial structures. The theory has emerged over recent decades as essential both for the analysis of algorithms and for the study of scientific models in other discliplines, including statistical physics, computational biology, and information theory.

What's next?

Suggestions for further study in Analytic Combinatorics

- Additional constructions and associated symbolic transfers
- Applications to paths in lattices and many other types
- Details of SA proofs
- Periodicity, irreducibility, algebraic functions
- Additional schema
- Drmota-Llaley-Woods theorem
- Technical conditions for SP approximations
- Multivariate asymptotics and limit laws
- Applications, applications, applications, applications



Available as "postscript" to this course

For an overview of Flajolet's work and current research in AC, watch the lecture "If you can specify it you can analyze it": the lasting legacy of Philippe Flajolet

Shameless plugs





Now this is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning. "

— Winston Churchill, 1942



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8. Saddle-Point Asymptotics

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